Chaos Theory Extended

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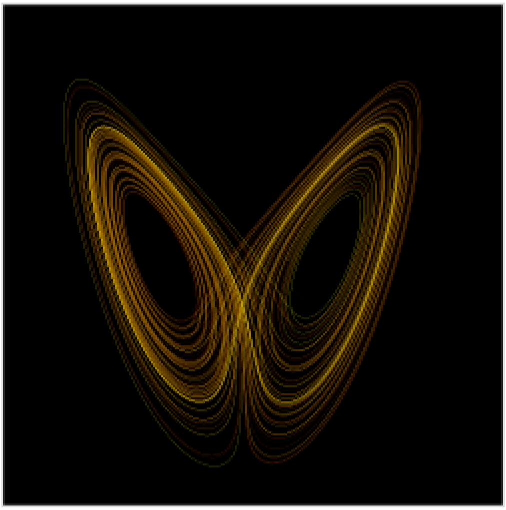
Subject classification: this is a mathematics resource.

Type classification: this resource is a learning project.

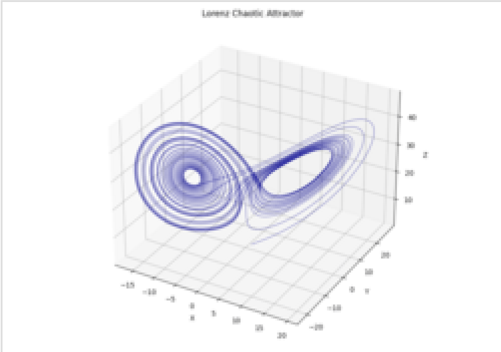
Chaos theory studies deterministic systems that can be predicted only for a limited period before their behavior begins to appear random. The length of time over which a chaotic system can be forecast reliably depends on three factors: the amount of uncertainty that can be tolerated in the prediction, the precision with which the system's current state can be measured, and a characteristic time scale determined by the system's dynamics, called the Lyapunov time. Examples of Lyapunov times include chaotic electrical circuits, around 1 millisecond; weather systems, a few days (though this is unproven); and the inner solar system, approximately 4 to 5 million years.[1] In chaotic systems, forecast uncertainty grows exponentially over time. Mathematically, doubling the forecast period increases the relative uncertainty by more than a factor of four. Consequently, in practice, reliable predictions cannot be made over periods longer than two or three Lyapunov times. When forecasts lose meaningful accuracy, the system's behavior effectively appears random.[2]



Plot of the Lorenz attractor with parameters $r = 28,\ \sigma = 10,\ b = \frac{8}{3}$.

Chaos in Everyday Life

Something "chaotic" in life, just means it seems random, uncontrollable, or impossible to predict. It's that feeling that a tiny event can blow up into something bigger.



A 3D visualization of the Lorenz attractor.

Everyday Examples of Chaos

- Traffic jams: One driver hits the brakes, and the whole highway slows to a crawl. A hesitation spreads and becomes a massive block.
- Weather: A small change, like the temperature being 0.1 °C higher in one spot, can shift the atmosphere enough to change next week's weather.
- People and relationships: One off timed remark can throw off a whole conversation or ruin a relationship.
  Being just five minutes late might mean missing someone who could have changed your life.

between our wish to control things and how complicated the world actually is.

Chaos in the Physical World

The universe follows exact laws, but chaos shows up everywhere, from the tiniest particles to everyday physics.

- Molecules in a Glass of Water A single glass of water contains $\pm 10^{23}$ molecules zipping around and colliding billions of times per second. Even if you somehow knew every molecule's exact location and speed, the smallest measurement error would grow so quickly that predictions would fall apart almost immediately. That's why physicists don't follow each particle, they use statistical mechanics instead.

Animation of a double-rod pendulum showing chaotic motion. Even tiny changes in the initial conditions lead to a dramatically different trajectory. The double pendulum is one of the simplest systems that displays chaotic behavior.

Turbulent Flow

If you turn a glass gently, the water flows smoothly. Turn it up, and the flow suddenly becomes messy and unpredictable. Smoke behaves the same way: a steady stream at first, then chaotic twisting. Fluid equations are completely deterministic, but nearly identical starting conditions can lead to wildly different results, classic chaos.

A Classic Example: The Double Pendulum

Set up two double pendulums with almost the same starting angle and speed, and within seconds they'll be swinging in totally different patterns. It's a simple mechanical device, but its behavior becomes unpredictable very quickly.

Fractals Everywhere

Coastlines, rivers, mountains, trees, blood vessels, broccoli—so many natural shapes repeat similar patterns at different scales. These fractal-like forms usually come from chaotic processes running over and over. Even heartbeats and brain signals show healthy chaotic variation; when they become too perfectly regular, that's usually a sign that something is wrong.

Chaos on Cosmic Scales

Even the universe at its largest scales is shaped by chaos.

The Early Universe

Right after the Big Bang, tiny quantum fluctuations created random differences in density. When the universe expanded rapidly during inflation, these tiny "wiggles" were stretched out and became the seeds of all future structure. The enormous web of galaxies and empty regions we see today grew out of that

Start with an almost even cloud of gas, add a few small density differences, and let gravity act for billions of years. With complications like explosions, black holes, and collisions, those tiny early variations decide whether a galaxy ends up as a smooth spiral or a jumbled merger full of starbursts.

Star Clusters

In dense star clusters, each star feels the pull of countless neighbors. The orbits are so sensitive that after just a few crossings, predictions break down. Stars get flung outward or drawn into the center, and the whole cluster slowly loses members because of this chaotic behavior.

Galaxy Collisions

When galaxies run into each other, stars almost never physically collide, but gravity throws them onto completely new paths. Long tidal tails, stretched bridges of stars, and distorted shapes appear. It’s chaotic but often stunning to look at. Yet even with all this underlying chaos, the universe still settles into recognizable patterns—spiral arms, predictable relations between galaxy properties, the smoothness of the cosmic microwave background, and more.

Bottom Line

Chaos isn’t the absence of order, it’s what helps create the most interesting kinds of order. From the swirls in a cup of coffee to massive cosmic structures spanning hundreds of millions of light-years, tiny differences can lead to huge outcomes. Perfect prediction will always be out of reach, and that fundamental unpredictability is part of what makes the universe active, complex, and far more than a simple clockwork machine.

Chaos theory

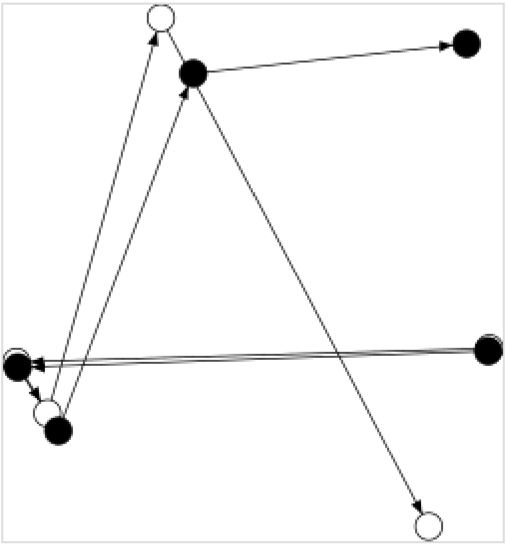
is an interdisciplinary field of study and a branch of mathematics that examines how deterministic dynamical systems can produce highly unpredictable behavior. Although such systems follow exact rules, they can respond extremely sensitively to their starting conditions, a property once mistaken for pure randomness.[3] Chaos theory shows that chaotic complex systems are not simply disordered. Hidden within their irregular behavior are repeating structures: patterns, interconnections, ongoing feedback loops, forms of self-similarity, fractal organization, and various kinds of self-organization.[4] A key concept is the butterfly effect: in a deterministic nonlinear system, a tiny shift in initial conditions can amplify into major changes later on.[5] This is often illustrated with the metaphor of a butterfly’s wings in Brazil influencing the development of a tornado in Texas.[6][7]:181–184[8] Small variations in starting conditions—whether from measurement inaccuracies or rounding in numerical computation—can cause systems of this kind to evolve in dramatically different ways, making reliable long-term forecasts of their behavior generally impossible.[9] This occurs even though these systems are deterministic—their future evolution is uniquely fixed by their initial state[10] and contains no random ingredients.[11] Thus, even

> Chaos: When the present determines the future but the approximate present does not approximately determine the future.

Chaotic dynamics appear in many natural contexts, such as fluid turbulence, irregular heart rhythms, and the behavior of weather and climate systems.[15][10] Chaos can also arise spontaneously in systems involving human-made components, like road traffic. Researchers analyze this behavior using chaotic mathematical models and tools such as recurrence plots and Poincaré maps. Chaos theory plays a role in diverse fields including meteorology,[10] anthropology,[16] sociology, environmental science, computer science, engineering, economics, ecology, and pandemic crisis management.[17][18] The theory has helped shape areas such as complex dynamical systems, edge of chaos theory, and self-assembly processes.

Chaotic dynamics

In everyday language, "chaos" refers to "a state of disorder".[19] In chaos theory, however, the term is used in a more precise sense. While there is no universally accepted mathematical definition of chaos, a widely used formulation by Robert L. Devaney states that a dynamical system is considered chaotic if it satisfies the following conditions:[20] it exhibits sensitivity to initial conditions, it is topologically transitive, it possesses dense periodic orbits. In some situations, the second and third properties can actually imply sensitivity to initial conditions.[21][22] For discrete-time systems, this holds for all continuous maps on metric spaces.[23] In these cases, although sensitivity to initial conditions is often the most practically relevant property, it does not need to be explicitly stated in the definition. When attention is restricted to intervals, the second property alone implies the other two.[24] A slightly weaker alternative definition of chaos uses only the first two conditions above.[25]

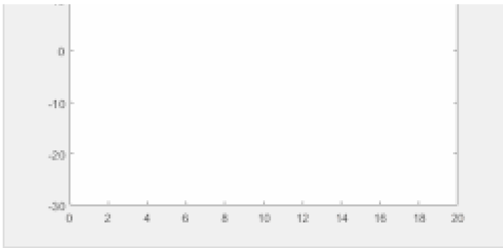


The map defined by $x \to 4x(1-x)$ and $y \to (x+y)$ mod 1 shows sensitivity to initial x positions. Two series of x and y values, starting from a very small difference, diverge rapidly over time.

Sensitivity to initial conditions

Main resource: Butterfly effect

Sensitivity to initial conditions refers to the property that in a chaotic system, points that are initially very close can follow drastically different trajectories over time. Even an imperceptibly small change in the starting state can lead to markedly different outcomes. This phenomenon is popularly known as the "butterfly effect", named after a 1972 presentation by Edward Lorenz to the American Association for the Advancement of Science in Washington, D.C., entitled *Predictability: Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?*.[26] In this metaphor, the butterfly’s wing represents a tiny trajectory could have been entirely different. As Lorenz emphasized in his 1993 book *The Essence of Chaos*,[7]:8 "sensitive dependence can serve as an acceptable definition of chaos". He defined the butterfly effect as: "The phenomenon that a small alteration in the state of a dynamical system will cause subsequent states to differ greatly from the states that would have followed without the alteration."[7]:23 This description matches the concept of sensitive dependence on initial conditions (SDIC). Lorenz illustrated SDIC with an idealized skiing model, showing how small changes in starting positions affect time-dependent paths.[7]:189–204 A predictability horizon can be estimated before SDIC fully manifests, i.e., before initially close trajectories diverge significantly.[27] A consequence of sensitivity to initial conditions is that if we start with a limited amount of information about the system (as is usually the case in practice), then beyond a certain time, the system would no longer be predictable. This is most prevalent in the case of weather, which is generally predictable only about a week ahead.[28] This does not mean that one cannot assert anything about events far in the future – only that some restrictions on the system are present. For example, we know that the temperature of the surface of the earth will not naturally reach 100 °C (212 °F) or fall below −130 °C (−202 °F) on earth (during the current geologic era), but we cannot predict exactly which day will have the hottest temperature of the year. In more mathematical terms, the Lyapunov exponent measures the sensitivity to initial conditions, in the form of rate of exponential divergence from the perturbed initial conditions.[29] More specifically, given two starting trajectories in the phase space that are infinitesimally close, with initial separation $\delta \mathbf{Z}_0$, the two trajectories end up diverging at a rate given by



Lorenz equations used to generate plots for the y variable. The initial conditions for x and z were kept the same, while those for y were slightly varied between 1.001, 1.0001, and 1.00001. The parameters were $\rho = 45.91$, $\sigma = 16$, and $\beta = 4$. The graph shows that even minimal differences in initial values produce substantial divergence after about 12 seconds.

$$|\delta \mathbf{Z}(t)| \approx e^{\lambda t} |\delta \mathbf{Z}_0|,$$

where t is the time and λ is the Lyapunov exponent. The rate of separation depends on the orientation of the initial separation vector, so a whole spectrum of Lyapunov exponents can exist. The number of Lyapunov exponents is equal to the number of dimensions of the phase space, though it is common to just refer to the largest one. For example, the maximal Lyapunov exponent (MLE) is most often used, because it determines the overall predictability of the system. A positive MLE, coupled with the solution's boundedness, is usually taken as an indication that the system is chaotic.[10] In addition to the above property, other properties related to sensitivity of initial conditions also exist. These include, for example, measure-theoretical mixing (as discussed in ergodic theory) and properties of a K-system.[13]

Non-periodicity

A chaotic system may have sequences of values for the evolving variable that exactly repeat themselves, giving periodic behavior starting from any point in that sequence. However, such periodic sequences are repelling rather than attracting, meaning that if the evolving variable is outside the sequence, however close, it will not enter the sequence and in fact, will diverge from it. Thus for almost all initial conditions, generally reliable for only about a week. This does not mean that nothing can be said about the distant future; rather, only broad constraints apply. Other mathematical notions related to sensitivity to initial conditions include measure-theoretic mixing from ergodic theory and properties associated with K-systems.[13] Chaotic systems may contain strictly periodic orbits, but these are repelling rather than attracting. A trajectory starting even slightly off such an orbit will move away from it over time. Consequently, for almost all initial conditions, the system evolves in a non-periodic, chaotic manner.

Combinatorial (or complex) chaos

Some definitions of chaos do not rely on sensitivity to initial conditions. One example is combinatorial chaos, which arises when a discrete combinatorial rule is applied repeatedly.[30] This kind of behavior is closely related to the dynamics seen in cellular automata. It is significant because systems exhibiting this form of chaos can be computationally universal: they can simulate a Turing machine, meaning that the halting problem becomes undecidable within their evolution. As a result, certain computational processes within such systems may never terminate. This represents a fundamentally different pathway to unpredictability.[31]

History

Barnsley fern created using the chaos game. Natural forms (ferns, clouds, mountains, etc.) may be recreated through an iterated function system (IFS).

James Clerk Maxwell was one of the first scientists to emphasize the importance of initial conditions, and he is considered an early contributor to chaos theory, with work in the 1860s and 1870s.[32][33][34] In the 1880s, while studying the three-body problem, Henri Poincaré discovered that certain orbits can be nonperiodic, yet neither diverge to infinity nor approach a fixed point.[35][36][37] In 1898, Jacques Hadamard published a study of a free particle moving frictionlessly on a surface of constant negative curvature, known as "Hadamard's billiards". He showed that all trajectories are unstable,[38] with particle trajectories diverging exponentially, corresponding to a positive Lyapunov exponent.[39] Further work on nonlinear differential equations was conducted by George David Birkhoff,[40] Andrey Nikolaevich Kolmogorov,[41][42][43] Mary Lucy Cartwright and John Edensor Littlewood,[44] and Stephen Smale.[45] Experimentalists and mathematicians had observed turbulence in fluid motion, chaotic behaviour in society and economy, nonperiodic oscillation in radio circuits, and fractal patterns in nature long before a formal theory existed.

A popular but inaccurate analogy for chaos

The sensitive dependence on initial conditions (i.e., the butterfly effect) has often been illustrated through the well-known piece of folklore:

For want of a rider, the battle was lost.
For want of a battle, the kingdom was lost.
And all for the want of a horseshoe nail.

Because of this verse, many readers incorrectly assume that the effect of a tiny initial perturbation must increase monotonically with time, or that any arbitrarily small change will inevitably produce a large impact in numerical integrations. In 2008, however, Lorenz argued that the verse does not describe true chaos, but instead illustrates the simpler notion of instability. The rhyme also suggests an irreversible cascade of consequences, whereas chaotic systems often exhibit later events that can partially offset earlier divergences. In this sense, the verse indicates divergence but omits the requirement of boundedness, which is necessary for the finite extent of a butterfly-shaped attractor. The behaviour described by the rhyme is therefore better characterized as "finite-time sensitive dependence."

Applications

A conus textile shell, similar in appearance to Rule 30, a cellular automaton with chaotic behaviour[46]

Although chaos theory originated from the study of weather, it has since found application in a broad range of fields. These include geology, mathematics, biology, computer science, economics,[47][48][49] engineering,[50][51] finance,[52][53][54][55][56] meteorology, philosophy, anthropology,[16] physics,[57][58][59] politics,[60][61] population dynamics,[62] and robotics. The following subsections provide examples but are not exhaustive, as new applications continue to emerge.

Cryptography

Main resource: Chaotic cryptology

Chaos theory has been used in cryptography for decades. In recent years, chaos and nonlinear dynamics have inspired hundreds of cryptographic primitives, including image encryption algorithms, hash functions, secure pseudorandom number generators, stream ciphers, watermarking, and steganography.[63] Most such algorithms use uni-modal chaotic maps, with the control parameters and initial conditions serving as cryptographic keys.[64] The conceptual similarity between chaotic maps and cryptographic systems is a major motivation for chaos-based design.[63] Symmetric-key cryptography relies on diffusion and confusion, which can be modeled effectively by chaotic dynamics.[65] In addition, the combination of chaos theory with DNA computing has been explored for image and data encryption,[66] although many DNA–chaos encryption schemes have later been shown insecure or inefficient.[67][68][69]

behaviour has also been observed in passive-dynamics biped robots, which can exhibit complex gait patterns.[71]

Biology

For more than a century, biologists have modeled species populations using population models, many of them continuous. More recently, chaotic models have been applied to certain discrete populations.[72] For example, time-series models of Canadian lynx populations have displayed evidence of chaotic dynamics.[73] Chaos is also investigated in ecological systems such as hydrology. Although hydrological models may face limitations, analyzing them from a chaotic perspective can still provide insight.[74] In cardiotocography, chaos-based modeling has been used to develop more sensitive indicators of fetal hypoxia while maintaining non-invasiveness.[75] As Perry notes, modeling chaotic time series in ecology benefits from appropriate constraints.[76]:176,177 Distinguishing genuine chaos from model-induced chaos can be difficult, so constrained models or duplicate time series help ensure realism, for instance in Perry & Wall 1984.[76]:176,177 In evolutionary biology, gene-for-gene co-evolution may exhibit chaotic dynamics in allele frequencies.[77] Adding variables, reflecting more realistic population structure, often increases the likelihood of chaos.[77] Foundational co-evolutionary studies by Robert M. May helped establish this line of research.[77] Even in constant environments, a single crop interacting with a single pathogen may generate quasi-periodic or chaotic oscillations in pathogen population.[78]

Economics

Economic models may also benefit from ideas from chaos theory, though assessing the stability of an economic system and identifying the most influential factors remains highly complex.[79] Unlike classical physical systems, economic and financial systems are fundamentally stochastic, emerging from interactions among people. Purely deterministic models therefore tend to fall short in representing economic data. Empirical attempts to test for chaos in economics and finance have produced mixed results, partly because studies sometimes confuse tests for genuine chaos with more general tests for nonlinear structure.[80] Chaos in economic time series can be investigated using recurrence quantification analysis (RQA). Orlando et al.[81] used the recurrence quantification correlation index to detect subtle structural changes in time-series data. The same technique has been applied to identify transitions from laminar (regular) to turbulent (chaotic) behavior, and to distinguish dynamical differences among macroeconomic variables, thereby exposing hidden features of economic evolution.[82] More recently, chaos-based approaches have been explored in modeling economic activity and incorporating shocks from external events such as COVID-19.[83]

See also

Examples of chaotic systems

- Advected contours
- Arnold's cat map
- Bifurcation theory
- Bouncing ball dynamics

- Double pendulum
- Duffing equation
- Dynamical billiards
- Economic bubble
- Gaspard-Rice system
- List of chaotic maps
- Rössler attractor
- Standard map
- Swinging Atwood's machine
- Tilt A Whirl

Other related topics

- Amplitude death
- Anosov diffeomorphism
- Catastrophe theory
- Causality
- Chaos as topological supersymmetry breaking
- Chaos machine
- Chaotic mixing
- Chaotic scattering
- Control of chaos
- Determinism
- Edge of chaos
- Emergence
- Mandelbrot set
- Kolmogorov–Arnold–Moser theorem
- Ill-conditioning
- Ill-posedness
- Nonlinear system
- Patterns in nature
- Predictability
- Quantum chaos
- Santa Fe Institute
- Shadowing lemma
- Synchronization of chaos
- Unintended consequence

People

- Ralph Abraham
- Michael Berry
- Leon O. Chua
- Ivar Ekeland
- Doyne Farmer
- Martin Gutzwiller
- Brosl Hasslacher
- Michel Hénon
- Aleksandr Lyapunov
- Norman Packard
- Otto Rössler
- David Ruelle
- Oleksandr Mikolaiovich Sharkovsky
- Greg Sams
- Robert Shaw
- Floris Takens
- James A. Yorke
- George M. Zaslavsky

Further reading

Articles

- Sharkovskii, A.N. (1964). "Co-existence of cycles of a continuous mapping of the line into itself". *Ukrainian Math. J.* **16**: 61–71.
- Li, T.Y.; Yorke, J.A. (1975). "Period Three Implies Chaos". *American Mathematical Monthly* **82** (10): 985–92. doi:10.2307/2318254. http://pb.math.univ.gda.pl/chaos/pdf/li-yorke.pdf.
- Alemansour, Hamed; Miandoab, Ehsan Maani; Pishkenari, Hossein Nejat (March 2017). "Effect of size on the chaotic behavior of nano resonators". *Communications in Nonlinear*

56 (8): 1242–57. doi:10.1007/s11253-005-0055-4.
- Day, R.H.; Pavlov, O.V. (2004). "Computing Economic Chaos". *Computational Economics* **23** (4): 289–301. doi:10.1023/B:CSEM.0000026787.81469.1f.
- Strelioff, C.; Hübler, A. (2006). "Medium-Term Prediction of Chaos". *Phys. Rev. Lett.* **96** (4). doi:10.1103/PhysRevLett.96.044101. 044101. PMID 16486826.
- Hübler, A.; Foster, G.; Phelps, K. (2007). "Managing Chaos: Thinking out of the Box". *Complexity* **12** (3): 10–13. doi:10.1002/cplx.20159.
- Motter, Adilson E.; Campbell, David K. (2013). "Chaos at fifty". *Physics Today* **66** (5): 27. doi:10.1063/PT.3.1977.

Textbooks

- Alligood, K.T.; Sauer, T.; Yorke, J.A. (1997). *Chaos: an introduction to dynamical systems*. Springer-Verlag. ISBN 978-0-387-94677-1. https://books.google.com/books?id=48YHnbHGZAgC.
- Baker, G. L. (1996). *Chaos, Scattering and Statistical Mechanics*. Cambridge University Press. ISBN 978-0-521-39511-3.
- Badii, R.; Politi A. (1997). *Complexity: hierarchical structures and scaling in physics*. Cambridge University Press. ISBN 978-0-521-66385-4. http://www.cambridge.org/gb/academic/subjects/physics/statistical-physics/complexity-hierarchical-structures-and-scaling-physics.
- Collet, Pierre; Jean-Pierre Eckmann (1980). *Iterated Maps on the Interval as Dynamical Systems*. Birkhauser. ISBN 978-0-8176-4926-5.
- Devaney, Robert L. (2003). *An Introduction to Chaotic Dynamical Systems* (2nd ed.). Westview Press. ISBN 978-0-8133-4085-2.
- Robinson, Clark (1995). *Dynamical systems: Stability, symbolic dynamics, and chaos*. CRC Press. ISBN 0-8493-8493-1.
- Feldman, D. P. (2012). *Chaos and Fractals: An Elementary Introduction*. Oxford University Press. ISBN 978-0-19-956644-0. https://dpfeldman.github.io/Chaos/index.html.
- Gollub, J. P.; Baker, G. L. (1996). *Chaotic dynamics*. Cambridge University Press. ISBN 978-0-521-47685-0. https://books.google.com/books?id=n1qnekRPKtoC.
- Guckenheimer, John; Holmes, Philip (1983). *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag. ISBN 978-0-387-90819-9.
- Gulick, Denny (1992). *Encounters with Chaos*. McGraw-Hill. ISBN 978-0-07-025203-5.
- Gutzwiller, Martin (1990). *Chaos in Classical and Quantum Mechanics*. Springer-Verlag. ISBN 978-0-387-97173-5. https://books.google.com/books?id=fnO3XYYpU54C.
- Hoover, William Graham (2001). *Time Reversibility, Computer Simulation, and Chaos*. World Scientific. ISBN 978-981-02-4073-8. https://books.google.com/books?id=24kEKsdl0psC.
- Kautz, Richard (2011). *Chaos: The Science of Predictable Random Motion*. Oxford University Press. ISBN 978-0-19-959458-0. https://books.google.com/books?id=x5YbNZjulN0C.
- Kiel, L. Douglas; Elliott, Euel W. (1997). *Chaos Theory in the Social Sciences*. Perseus Publishing. ISBN 978-0-472-08472-2. https://books.google.com/books?id=K46kkMXnKfcC.
- Moon, Francis (1990). *Chaotic and Fractal Dynamics*. Springer-Verlag. ISBN 978-0-471-54571-2. https://books.google.com/books?id=Ddz-CI-nSKYC.

- Ott, Edward (2002). *Chaos in Dynamical Systems*. Cambridge University Press. ISBN 978-0-521-01084-9. https://books.google.com/books?id=nOLx--zzHSgC.
- Strogatz, Steven (2000). *Nonlinear Dynamics and Chaos*. Perseus Publishing. ISBN 978-0-7382-0453-6. https://archive.org/details/nonlineardynamic00stro.
- Sprott, Julien Clinton (2003). *Chaos and Time-Series Analysis*. Oxford University Press. ISBN 978-0-19-850840-3. https://books.google.com/books?id=SEDjdjPZ158C.
- Tél, Tamás; Gruiz, Márton (2006). *Chaotic dynamics: An introduction based on classical mechanics*. Cambridge University Press. ISBN 978-0-521-83912-9. https://books.google.com/books?id=P2JL7s2IvakC.
- Teschl, Gerald (2012). *Ordinary Differential Equations and Dynamical Systems*. Providence: American Mathematical Society. ISBN 978-0-8218-8328-0. https://www.mat.univie.ac.at/~gerald/ftp/book-ode/.
- *Nonlinear Dynamics And Chaos*. John Wiley and Sons Ltd. 2001. ISBN 978-0-471-87645-8.
- Tufillaro; Reilly (1992). *An experimental approach to nonlinear dynamics and chaos*. American Journal of Physics. **61**. Addison-Wesley. p. 958. doi:10.1119/1.17380. ISBN 978-0-201-55441-0. Bibcode: 1993AmJPh..61..958T.
- Wiggins, Stephen (2003). *Introduction to Applied Dynamical Systems and Chaos*. Springer. ISBN 978-0-387-00177-7.
- Zaslavsky, George M. (2005). *Hamiltonian Chaos and Fractional Dynamics*. Oxford University Press. ISBN 978-0-19-852604-9.

Semitechnical and popular works

- Christophe Letellier, *Chaos in Nature*, World Scientific Publishing Company, 2012, ISBN 978-981-4374-42-2.
- Abraham, Ralph H.; Ueda, Yoshisuke, eds (2000). *The Chaos Avant-Garde: Memoirs of the Early Days of Chaos Theory*. World Scientific Series on Nonlinear Science Series A. **39**. World Scientific. doi:10.1142/4510. ISBN 978-981-238-647-2. Bibcode: 2000cagm.book.....A. https://books.google.com/books?id=0E667XpBq1UC.
- Barnsley, Michael F. (2000). *Fractals Everywhere*. Morgan Kaufmann. ISBN 978-0-12-079069-2. https://books.google.com/books?id=oh7NoePgmOIC.
- Bird, Richard J. (2003). *Chaos and Life: Complexity and Order in Evolution and Thought*. Columbia University Press. ISBN 978-0-231-12662-5. https://books.google.com/books?id=fv3sltQBS54C.
- John Briggs and David Peat, *Turbulent Mirror: An Illustrated Guide to Chaos Theory and the Science of Wholeness*, Harper Perennial 1990, 224 pp.
- John Briggs and David Peat, *Seven Life Lessons of Chaos: Spiritual Wisdom from the Science of Change*, Harper Perennial 2000, 224 pp.
- Cunningham, Lawrence A. (1994). "From Random Walks to Chaotic Crashes: The Linear Genealogy of the Efficient Capital Market Hypothesis". *George Washington Law Review* **62**: 546.
- Predrag Cvitanović, *Universality in Chaos*, Adam Hilger 1989, 648 pp.
- Leon Glass and Michael C. Mackey, *From Clocks to Chaos: The Rhythms of Life,* Princeton University Press 1988, 272 pp.
- James Gleick, *Chaos: Making a New Science*, New York: Penguin, 1988. 368 pp.
- W:John Gribbin. *Deep Simplicity*. Penguin Press Science. Penguin Books.

- Hans Lauwerier, *Fractals*, Princeton University Press, 1991.
- Edward Lorenz, *The Essence of Chaos*, University of Washington Press, 1996.
- Marshall, Alan (2002). *The Unity of Nature - Wholeness and Disintegration in Ecology and Science*. doi:10.1142/9781860949548. ISBN 978-1-86094-954-8.
- David Peak and Michael Frame, *Chaos Under Control: The Art and Science of Complexity*, Freeman, 1994.
- Heinz-Otto Peitgen and Dietmar Saupe (Eds.), *The Science of Fractal Images*, Springer 1988, 312 pp.
- Nuria Perpinya, *Caos, virus, calma. La Teoría del Caos aplicada al desórden artístico, social y político*, Páginas de Espuma, 2021.
- Clifford A. Pickover, *Computers, Pattern, Chaos, and Beauty: Graphics from an Unseen World* , St Martins Pr 1991.
- Clifford A. Pickover, *Chaos in Wonderland: Visual Adventures in a Fractal World*, St Martins Pr 1994.
- Ilya Prigogine and Isabelle Stengers, *Order Out of Chaos*, Bantam 1984.
- Peitgen, Heinz-Otto; Richter, Peter H. (1986). *The Beauty of Fractals*. doi:10.1007/978-3-642-61717-1. ISBN 978-3-642-61719-5. https://archive.org/details/beautyoffractals0000peit.
- David Ruelle, *Chance and Chaos*, Princeton University Press 1993.
- Ivars Peterson, *Newton's Clock: Chaos in the Solar System*, Freeman, 1993.
- W:Ian Roulstone; W:John Norbury (2013). *Invisible in the Storm: the role of mathematics in understanding weather*. Princeton University Press. ISBN 978-0-691-15272-1. https://books.google.com/books?id=qnMrFEHMrWwC.
- Ruelle, D. (1989). *Chaotic Evolution and Strange Attractors*. doi:10.1017/CBO9780511608773. ISBN 978-0-521-36272-6. https://archive.org/details/chaoticevolution0000ruel.
- Manfred Schroeder, *Fractals, Chaos, and Power Laws*, Freeman, 1991.
- Smith, Peter (1998). *Explaining Chaos*. doi:10.1017/CBO9780511554544. ISBN 978-0-511-55454-4.
- Ian Stewart, *Does God Play Dice?: The Mathematics of Chaos* , Blackwell Publishers, 1990.
- Steven Strogatz, *Sync: The emerging science of spontaneous order*, Hyperion, 2003.
- Yoshisuke Ueda, *The Road To Chaos*, Aerial Pr, 1993.
- M. Mitchell Waldrop, *Complexity : The Emerging Science at the Edge of Order and Chaos*, Simon & Schuster, 1992.
- Antonio Sawaya, *Financial Time Series Analysis : Chaos and Neurodynamics Approach*, Lambert, 2012.

External links

- Nonlinear Dynamics Research Group (https://web.archive.org/web/20160310065017/http://lagrange.physics.drexel.edu/) with Animations in Flash
- The Chaos group at the University of Maryland (http://www.chaos.umd.edu/)
- The Chaos Hypertextbook (https://hypertextbook.com/chaos/). An introductory primer on chaos and fractals

- Nonlinear Dynamics Research Group at CSDC (https://web.archive.org/web/20130425220329/http://www.csdc.unifi.it/mdswitch.html?newlang=eng), Florence, Italy
- Nonlinear dynamics: how science comprehends chaos (http://www.creatingtechnology.org/papers/chaos.htm), talk presented by Sunny Auyang, 1998.
- Nonlinear Dynamics (https://www.egwald.ca/nonlineardynamics/). Models of bifurcation and chaos by Elmer G. Wiens
- Gleick's *Chaos* (excerpt) (https://around.com/books/chaos/) Archived (https://web.archive.org/web/20070202075958/http://www.around.com/chaos.html) 2007-02-02 at the Wayback Machine
- Systems Analysis, Modelling and Prediction Group (https://web.archive.org/web/20070428110552/http://www.eng.ox.ac.uk/samp/) at the University of Oxford
- A page about the Mackey-Glass equation (https://web.archive.org/web/20090307094012/http://www.mgix.com/snippets/?MackeyGlass)
- High Anxieties — The Mathematics of Chaos (https://www.youtube.com/watch?v=5pKrKdNclYs) (2008) BBC documentary directed by David Malone
- The chaos theory of evolution (https://www.newscientist.com/article/mg20827821.000-the-chaos-theory-of-evolution.html) – article published in Newscientist featuring similarities of evolution and non-linear systems including fractal nature of life and chaos.
- Jos Leys, Étienne Ghys et Aurélien Alvarez, *Chaos, A Mathematical Adventure* (https://www.chaos-math.org/en.html). Nine films about dynamical systems, the butterfly effect and chaos theory, intended for a wide audience.
- "Chaos Theory" (https://www.bbc.co.uk/programmes/p00548f6), BBC Radio 4 discussion with Susan Greenfield, David Papineau & Neil Johnson (*In Our Time*, May 16, 2002)
- Chaos: The Science of the Butterfly Effect (https://www.youtube.com/watch?v=fDek6cYijxI) (2019) an explanation presented by Derek Muller

Attribution

References

1. Wisdom, Jack; Sussman, Gerald Jay (1992-07-03). "Chaotic Evolution of the Solar System". *Science* **257** (5066): 56–62. doi:10.1126/science.257.5066.56. ISSN 1095-9203. PMID 17800710.
2. *Sync: The Emerging Science of Spontaneous Order*, Steven Strogatz, Hyperion, New York, 2003, pages 189–190.
3. "Chaos Theory: Definition & Facts". *Encyclopedia Britannica*. Retrieved 2019-11-24.
4. "What Is Chaos Theory? – Fractal Foundation". *Fractal Foundation*. Retrieved 2019-11-24.
5. Weisstein, Eric W. "Chaos". *MathWorld*. Wolfram Research. Retrieved 2019-11-24.
6. Boeing, Geoff (2015-03-26). "Chaos Theory and the Logistic Map". Retrieved 2020-05-17.
7. Lorenz, Edward (1993). *The Essence of Chaos*. University of Washington Press. ISBN 978-0-295-97514-6. https://books.google.com/books?id=j5Ub6sMCoOsC.
8. Shen, Bo-Wen; Pielke, Roger A.; Zeng, Xubin; Cui, Jialin; Faghih-Naini, Sara; Paxson, Wei; Atlas, Robert (2022-07-04). "Three Kinds of Butterfly Effects within Lorenz Models". *Encyclopedia* **2** (3): 1250–1259. doi:10.3390/encyclopedia2030084. ISSN 2673-8392.

https://archive.org/details/inwakeofchaosunp0000kell.

10. Bishop, Robert (2017), "Chaos", in Zalta, Edward N. (ed.), *The Stanford Encyclopedia of Philosophy* (Spring 2017 ed.), Metaphysics Research Lab, Stanford University, retrieved 2019-11-24
11. Kellert, Stephen H. (1993). *In the Wake of Chaos: Unpredictable Order in Dynamical Systems*. University of Chicago Press. p. 56. ISBN 978-0-226-42976-2. https://archive.org/details/inwakeofchaosunp0000kell/page/56.
12. Kellert, Stephen H. (1993). *In the Wake of Chaos: Unpredictable Order in Dynamical Systems*. University of Chicago Press. p. 62. ISBN 978-0-226-42976-2. https://archive.org/details/inwakeofchaosunp0000kell/page/62.
13. Werndl, Charlotte (2009). "What are the New Implications of Chaos for Unpredictability?". *The British Journal for the Philosophy of Science* **60** (1): 195–220. doi:10.1093/bjps/axn053.
14. Danforth, Christopher M. (April 2013). "Chaos in an Atmosphere Hanging on a Wall". *Mathematics of Planet Earth 2013*. Retrieved 12 June 2018.
15. Ivancevic, Vladimir G.; Tijana T. Ivancevic (2008). *Complex nonlinearity: chaos, phase transitions, topology change, and path integrals*. Springer. ISBN 978-3-540-79356-4.
16. Mosko M.S., Damon F.H. (Eds.) (2005). *On the order of chaos. Social anthropology and the science of chaos*. Oxford: Berghahn Books.
17. Piotrowski, Chris. "Covid-19 Pandemic and Chaos Theory: Applications based on a Bibliometric Analysis". *researchgate.net*. Retrieved 2020-05-13.
18. Weinberger, David (2019). *Everyday Chaos – Technology, Complexity and How We're Thriving in a New World of Possibility*. Harvard Business Review Press. ISBN 978-1-63369-396-8. https://books.google.com/books?id=R7V2DwAAQBAJ.
19. "Definition of chaos | Dictionary.com". *www.dictionary.com*. Retrieved 2019-11-24.
20. Hasselblatt, Boris; Anatole Katok (2003). *A First Course in Dynamics: With a Panorama of Recent Developments*. Cambridge University Press. ISBN 978-0-521-58750-1.
21. Elaydi, Saber N. (1999). *Discrete Chaos*. Chapman & Hall/CRC. p. 137. ISBN 978-1-58488-002-8.
22. Basener, William F. (2006). *Topology and its applications*. Wiley. p. 42. ISBN 978-0-471-68755-9.
23. Banks; Brooks; Cairns; Davis; Stacey (1992). "On Devaney's definition of chaos". *The American Mathematical Monthly* **99** (4): 332–334. doi:10.1080/00029890.1992.11995856.
24. Vellekoop, Michel; Berglund, Raoul (April 1994). "On Intervals, Transitivity = Chaos". *The American Mathematical Monthly* **101** (4): 353–5. doi:10.2307/2975629.
25. Medio, Alfredo; Lines, Marji (2001). *Nonlinear Dynamics: A Primer*. Cambridge University Press. p. 165. ISBN 978-0-521-55874-7. https://archive.org/details/nonlineardynamic00medi.
26. "Edward Lorenz, father of chaos theory and butterfly effect, dies at 90". *MIT News*. 16 April 2008. Retrieved 2019-11-24.
27. Shen, Bo-Wen; Pielke, Roger A.; Zeng, Xubin (2022-05-07). "One Saddle Point and Two Types of Sensitivities within the Lorenz 1963 and 1969 Models". *Atmosphere* **13** (5): 753. doi:10.3390/atmos13050753. ISSN 2073-4433.
28. Watts, Robert G. (2007). *Global Warming and the Future of the Earth*. Morgan & Claypool. p. 17. https://archive.org/details/globalwarmingfut00watt_399.
29. Weisstein, Eric W. "Lyapunov Characteristic Exponent". *mathworld.wolfram.com*. Retrieved 2019-11-24.
30. "Science: Mathematician discovers a more complex form of chaos".

33. Everitt, Francis (2006-12-01). "James Clerk Maxwell: a force for physics". *Physics World*. Retrieved 2023-11-03.
34. Gardini, Laura; Grebogi, Celso; Lenci, Stefano (2020-10-01). "Chaos theory and applications: a retrospective on lessons learned and missed or new opportunities". *Nonlinear Dynamics* **102** (2): 643–644. doi:10.1007/s11071-020-05903-0. ISSN 1573-269X.
35. Poincaré, Jules Henri (1890). "Sur le problème des trois corps et les équations de la dynamique. Divergence des séries de M. Lindstedt". *Acta Mathematica* **13** (1–2): 1–270. doi:10.1007/BF02392506.
36. Poincaré, J. Henri (2017). *The three-body problem and the equations of dynamics: Poincaré's foundational work on dynamical systems theory*. Popp, Bruce D. (Translator). Cham, Switzerland: Springer International Publishing. ISBN 978-3-319-52898-4. OCLC 987302273.
37. Diacu, Florin; Holmes, Philip (1996). *Celestial Encounters: The Origins of Chaos and Stability*. Princeton University Press. ISBN 978-0-691-02743-2.
38. Hadamard, Jacques (1898). "Les surfaces à courbures opposées et leurs lignes géodesiques". *Journal de Mathématiques Pures et Appliquées* **4**: 27–73. https://www.numdam.org/item/JMPA_1898_5_4__27_0.pdf.
39. Aurich, R.; Sieber, M.; Steiner, F. (1 August 1988). "Quantum Chaos of the Hadamard–Gutzwiller Model". *Physical Review Letters* **61** (5): 483–487. doi:10.1103/PhysRevLett.61.483. PMID 10039347. http://bib-pubdb1.desy.de/record/324018/files/PhysRevLett61483.pdf.
40. W:George D. Birkhoff, *Dynamical Systems,* vol. 9 of the American Mathematical Society Colloquium Publications (Providence, Rhode Island: American Mathematical Society, 1927)
41. Kolmogorov, A. N. (1941). "The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers". *Doklady Akademii Nauk SSSR* **30** (4): 301–305. Kolmogorov, A. N. (1991). "The Local Structure of Turbulence in Incompressible Viscous Fluid for Very Large Reynolds Numbers". *Proceedings of the Royal Society A* **434** (1890): 9–13. doi:10.1098/rspa.1991.0075.
42. Kolmogorov, A. N. (1941). "On degeneration of isotropic turbulence in an incompressible viscous liquid". *Doklady Akademii Nauk SSSR* **31** (6): 538–540. Reprinted in: Kolmogorov, A. N. (1991). "Dissipation of Energy in the Locally Isotropic Turbulence". *Proceedings of the Royal Society A* **434** (1890): 15–17. doi:10.1098/rspa.1991.0076.
43. Kolmogorov, A. N. (1979). "Preservation of conditionally periodic movements with small change in the Hamilton function". *Stochastic Behavior in Classical and Quantum Hamiltonian Systems*. Lecture Notes in Physics. **93**. pp. 51–56. doi:10.1007/BFb0021737. ISBN 978-3-540-09120-2. Bibcode: 1979LNP....93...51K. Translation of *Doklady Akademii Nauk SSSR* (1954) 98: 527.
44. Cartwright, Mary L.; Littlewood, John E. (1945). "On non-linear differential equations of the second order, I: The equation $y'' + k(1-y^2)y' + y = b\lambda k\cos(\lambda t + a)$, k large". *Journal of the London Mathematical Society* **20** (3): 180–9. doi:10.1112/jlms/s1-20.3.180.
45. Smale, Stephen (January 1960). "Morse inequalities for a dynamical system". *Bulletin of the American Mathematical Society* **66**: 43–49. doi:10.1090/S0002-9904-1960-10386-2.
46. Stephen Coombes (February 2009). "The Geometry and Pigmentation of Seashells" (PDF). University of Nottingham. Archived from the original (PDF) on 2013-11-05. Retrieved 2013-04-10.
47. Kyrtsou C.; Labys W. (2006). "Evidence for chaotic dependence between US inflation and commodity prices". *Journal of Macroeconomics* **28** (1): 256–266.

49. Kyrtsou, C.; Vorlow, C. (2005). "Complex dynamics in macroeconomics: A novel approach". *New Trends in Macroeconomics*. Springer Verlag.
50. Hernández-Acosta, M. A.; Trejo-Valdez, M.; Castro-Chacón, J. H.; Miguel, C. R. Torres-San; Martínez-Gutiérrez, H. (2018). "Chaotic signatures of photoconductive Cu 2 ZnSnS 4 nanostructures explored by Lorenz attractors". *New Journal of Physics* **20** (2): 023048. doi:10.1088/1367-2630/aaad41. ISSN 1367-2630.
51. "Applying Chaos Theory to Embedded Applications". Archived from the original on 9 August 2011.
52. Hristu-Varsakelis, D.; Kyrtsou, C. (2008). "Evidence for nonlinear asymmetric causality in US inflation, metal and stock returns". *Discrete Dynamics in Nature and Society* **2008**: 1–7. doi:10.1155/2008/138547. 138547.
53. Kyrtsou, C.; M. Terraza (2003). "Is it possible to study chaotic and ARCH behaviour jointly? Application of a noisy Mackey-Glass equation with heteroskedastic errors to the Paris Stock Exchange returns series". *Computational Economics* **21** (3): 257–276. doi:10.1023/A:1023939610962.
54. Gregory-Williams, Justine; Williams, Bill (2004). *Trading Chaos: Maximize Profits with Proven Technical Techniques* (2nd ed.). New York: Wiley. ISBN 978-0-471-46308-5.
55. Peters, Edgar E. (1994). *Fractal market analysis: applying chaos theory to investment and economics* (2. print. ed.). New York u.a.: Wiley. ISBN 978-0-471-58524-4.
56. Peters, Edgar E. (1996). *Chaos and order in the capital markets: a new view of cycles, prices, and market volatility* (2nd ed.). New York: John Wiley & Sons. ISBN 978-0-471-13938-6.
57. Hubler, A.; Phelps, K. (2007). "Guiding a self-adjusting system through chaos". *Complexity* **13** (2): 62. doi:10.1002/cplx.20204.
58. Gerig, A. (2007). "Chaos in a one-dimensional compressible flow". *Physical Review E* **75** (4). doi:10.1103/PhysRevE.75.045202. PMID 17500951.
59. Wotherspoon, T.; Hubler, A. (2009). "Adaptation to the Edge of Chaos in the Self-Adjusting Logistic Map". *The Journal of Physical Chemistry A* **113** (1): 19–22. doi:10.1021/jp804420g. PMID 19072712.
60. Borodkin, Leonid I. (2019). "Challenges of Instability: The Concepts of Synergetics in Studying the Historical Development of Russia". *Ural Historical Journal* **63** (2): 127–136. doi:10.30759/1728-9718-2019-2(63)-127-136.
61. Progonati, E (2018). "Brexit in the Light of Chaos Theory and Some Assumptions About the Future of the European Union". *Chaos, complexity and leadership 2018 explorations of chaotic and complexity theory*. Springer. ISBN 978-3-030-27672-0.
62. Dilão, R.; Domingos, T. (2001). "Periodic and Quasi-Periodic Behavior in Resource Dependent Age Structured Population Models". *Bulletin of Mathematical Biology* **63** (2): 207–230. doi:10.1006/bulm.2000.0213. PMID 11276524.
63. Akhavan, A.; Samsudin, A.; Akhshani, A. (2011-10-01). "A symmetric image encryption scheme based on combination of nonlinear chaotic maps". *Journal of the Franklin Institute* **348** (8): 1797–1813. doi:10.1016/j.jfranklin.2011.05.001.
64. Behnia, S.; Akhshani, A.; Mahmodi, H.; Akhavan, A. (2008-01-01). "A novel algorithm for image encryption based on mixture of chaotic maps". *Chaos, Solitons & Fractals* **35** (2): 408–419. doi:10.1016/j.chaos.2006.05.011.
65. Wang, Xingyuan; Zhao, Jianfeng (2012). "An improved key agreement protocol based on chaos". *Commun. Nonlinear Sci. Numer. Simul.* **15** (12): 4052–4057. doi:10.1016/j.cnsns.2010.02.014.

doi:10.1016/j.optlastec.2017.04.022.

68. Xu, Ming (2017-06-01). "Cryptanalysis of an Image Encryption Algorithm Based on DNA Sequence Operation and Hyper-chaotic System". *3D Research* **8** (2). doi:10.1007/s13319-017-0126-y. ISSN 2092-6731.
69. Liu, Yuansheng; Tang, Jie; Xie, Tao (2014-08-01). "Cryptanalyzing a RGB image encryption algorithm based on DNA encoding and chaos map". *Optics & Laser Technology* **60**: 111–115. doi:10.1016/j.optlastec.2014.01.015.
70. Nehmzow, Ulrich; Keith Walker (Dec 2005). "Quantitative description of robot–environment interaction using chaos theory". *Robotics and Autonomous Systems* **53** (3–4): 177–193. doi:10.1016/j.robot.2005.09.009. Archived from the original on 2017-08-12. http://cswww.essex.ac.uk/staff/udfn/ftp/ecmrw3.pdf. Retrieved 2017-10-25.
71. Goswami, Ambarish; Thuilot, Benoit; Espiau, Bernard (1998). "A Study of the Passive Gait of a Compass-Like Biped Robot: Symmetry and Chaos". *The International Journal of Robotics Research* **17** (12): 1282–1301. doi:10.1177/027836499801701202.
72. Eduardo, Liz; Ruiz-Herrera, Alfonso (2012). "Chaos in discrete structured population models". *SIAM Journal on Applied Dynamical Systems* **11** (4): 1200–1214. doi:10.1137/120868980.
73. Lai, Dejian (1996). "Comparison study of AR models on the Canadian lynx data: a close look at BDS statistic". *Computational Statistics & Data Analysis* **22** (4): 409–423. doi:10.1016/0167-9473(95)00056-9.
74. Sivakumar, B. (2000-01-31). "Chaos theory in hydrology: important issues and interpretations". *Journal of Hydrology* **227** (1–4): 1–20. doi:10.1016/S0022-1694(99)00186-9.
75. Bozóki, Zsolt (1997-02). "Chaos theory and power spectrum analysis in computerized cardiotocography". *European Journal of Obstetrics & Gynecology and Reproductive Biology* **71** (2): 163–168. doi:10.1016/s0301-2115(96)02628-0. PMID 9138960.
76. Perry, Joe N.; Smith, Robert H.; Woiwod, Ian P.; Morse, David R. (2000). *Chaos in Real Data: The Analysis of Non-Linear Dynamics from Short Ecological Time Series*. Population and Community Biology Series. **22** (1 ed.). Springer. doi:10.1007/978-94-011-4010-2. ISBN 978-94-010-5772-1.
77. Thompson, John N.; Burdon, Jeremy J. (1992). "Gene-for-gene coevolution between plants and parasites". *Nature* **360** (6400): 121–125. doi:10.1038/360121a0.
78. Jones, Gareth (1998). *The Epidemiology of Plant Diseases*. Springer. doi:10.1007/978-94-017-3302-1. ISBN 978-94-017-3302-1.
79. Juárez, Fernando (2011). "Applying the theory of chaos and a complex model of health to establish relations among financial indicators". *Procedia Computer Science* **3**: 982–986. doi:10.1016/j.procs.2010.12.161.
80. Brooks, Chris (1998). "Chaos in foreign exchange markets: a sceptical view". *Computational Economics* **11** (3): 265–281. doi:10.1023/A:1008650024944. ISSN 1572-9974.
81. Orlando, Giuseppe; Zimatore, Giovanna (18 December 2017). "RQA correlations on real business cycles time series". *Indian Academy of Sciences – Conference Series* **1** (1): 35–41. doi:10.29195/iascs.01.01.0009.
82. Orlando, Giuseppe; Zimatore, Giovanna (1 May 2018). "Recurrence quantification analysis of business cycles". *Chaos, Solitons & Fractals* **110**: 82–94. doi:10.1016/j.chaos.2018.02.032.
83. Orlando, Giuseppe; Zimatore, Giovanna (1 August 2020). "Business cycle modeling between financial crises and black swans: Ornstein–Uhlenbeck stochastic process vs Kaldor