Quantum A Spooky Action at a Distance

←Quantum

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Subject classification: this is a physics resource.

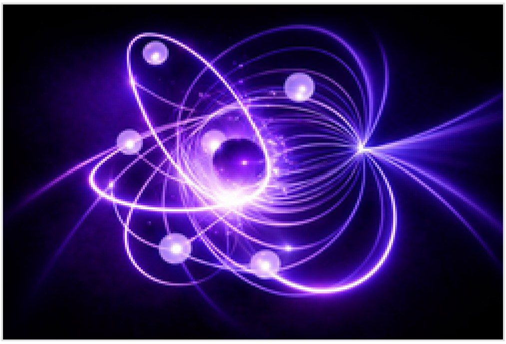
Type classification: this is a quiz resource.

Type classification: this resource is a learning project.

This page is an advanced conceptual overview of quantum entanglement intended for readers with prior coursework in quantum mechanics. It emphasizes conceptual structure rather than step-by-step instruction.

Quantum Entanglemen Course Overview

Welcome to the Wikiversity course on Quantum Entanglement. This course explores the fundamental concepts, history, mathematical details, and applications of quantum entanglement, a key phenomenon in quantum mechanics. Learning Objectives By the end of this course, learners will be able to:



Artistic impression of an atom 2b

- Understand the basic principles of quantum entanglement and its differences from classical correlations.
- Explain the historical development, including the EPR paradox and Bell's theorem.
- Describe mathematical formulations of entangled states.
- Discuss applications in quantum information and experiments demonstrating entanglement.

Prerequisites

Basic knowledge of quantum mechanics, linear algebra, and probability is recommended.

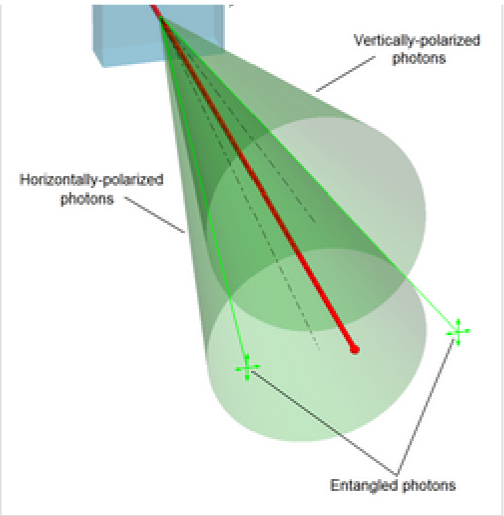
Course Structure

The course is divided into modules based on the provided content. Each module includes readings, key concepts, and optional discussion questions.

Module 1: Introduction to Quantum Entanglement

Quantum entanglement occurs when the quantum state of a composite system cannot be factored into independent states of its individual particles, regardless of the distance separating them. This behavior has no analogue in classical physics.[1]

down along that axis. The act of measurement projects the joint quantum state of the system into a definite outcome, meaning the entangled state applies to the system as a whole rather than to the individual particles.



Spontaneous parametric down-conversion process can split photons into type II photon pairs with mutually perpendicular polarization.

This weirdness kicked off with a 1935 paper from Albert Einstein, Boris Podolsky, and Nathan Rosen,[2] and some follow-ups from Erwin Schrödinger,[3][4] laying out what's now called the EPR paradox. Einstein and crew thought it was nuts because it messed with local realism's take on cause and effect[5] and figured quantum mechanics must be missing something.

But later, experiments proved quantum's predictions right, with polarization or spin measurements on distant entangled particles breaking Bell's inequality in stats.[6][7][8][9] You cannot explain these links with local hidden variables inside the particles. Still, even though entanglement creates these correlations over huge distances, you cannot use it to send messages faster than the speed of light.[10][11][12]:453 Quantum entanglement with photons,[13][14] electrons,[15][16] top quarks,[17] molecules[18] and even small diamonds.[19] The use of quantum entanglement in communication and computation is an area of research and development.

Key Concepts

- Definition of entanglement.
- EPR paradox and "spooky action at a distance."
- No faster-than-light communication.

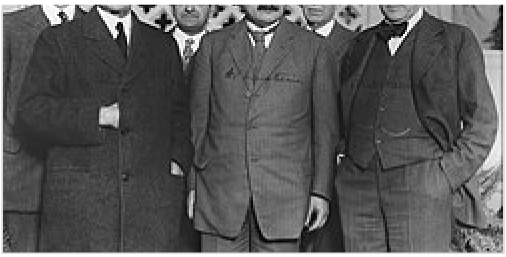
Discussion Question

How does quantum entanglement challenge classical intuitions about locality?

Module 2: History of Quantum Entanglement

Background: History of quantum mechanics

Albert Einstein and Niels Bohr got into a long, argument about how to interpret quantum mechanics, the Bohr-Einstein debates. Einstein came up with this thought experiment about a box with a photon, pointing out that what you measure at the box changes what you can tell about the photon over there. He worked this out by 1931, basically looking at what we later call entanglement.[20] That same year, Hermann Weyl wrote in his book on group theory and quantum mechanics that when parts of a system interact, the whole thing has this Gestalt quality, the whole's more than just the parts added up.[21][22] In 1932, Erwin Schrödinger figured out the key equations for entanglement but didn't publish them.[23] In



Portrait of Albert Einstein and Others

Rosen put out their paper on the EPR paradox, arguing that quantum mechanics' wave function doesn't give a complete picture of reality.[2] They talked about two systems that interact and then split apart, and after that, quantum mechanics can't describe them on their own.

Right after that paper dropped, Erwin Schrödinger wrote Einstein a letter in German, using the word "Verschränkung" (which he translated as "entanglement") for those EPR situations.[25] Schrödinger then wrote a full paper explaining entanglement,[26] calling it not just one but *the* key feature of quantum mechanics that sets it apart from classical thinking.[3]

Like Einstein, Schrödinger wasn't thrilled with entanglement, it seemed to break the relativity rule on how fast info can travel.[27] Einstein mocked it as "spukhafte Fernwirkung" or "spooky action at a distance," where measuring something here instantly sets a property over there.[28][29]

In 1946, John Archibald Wheeler suggested checking the polarization of gamma-ray photon pairs from electron-positron annihilation.[30] Chien-Shiung Wu and I. Shaknov did the experiment in 1949,[31] showing you could make EPR-type entangled pairs in a lab.[32] Even though Schrödinger called it crucial, not much got written about entanglement for years after his paper.[26] Then in 1964, John S. Bell showed there's a limit, Bell's inequality, on how strong correlations can be in any local realism theory, and quantum predicts breaking that for some entangled systems.[33][34]:405 You can test this, and experiments started with Stuart Freedman and John Clauser in 1972[6] and Alain Aspect in 1982.[35][36][37] Bell was not keen on students chasing this stuff, he thought it was too fringe, but after a lecture at Oxford, a student named Artur Ekert suggested using Bell inequality violations for communication.[38][1]:874

In 1992, academics started using entanglement to suggest quantum teleportation,[39] and have experimented this by 1997.[40][41][42] In 1990, Anton Zeilinger used parametric down-conversion to create entanglement, leading to entanglement swapping[43]:317 and showing quantum cryptography with entangled photons.[44][45] In 2022, the Nobel Prize in Physics went to Aspect, Clauser, and Zeilinger for their entangled photon experiments, proving Bell inequality violations, and starting quantum info science.[46]

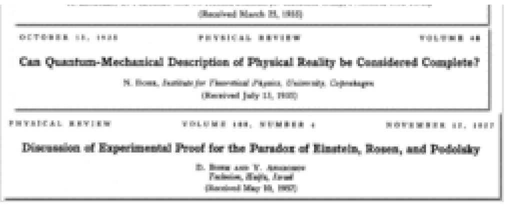
Key Concepts

- Bohr-Einstein debates.
- Contributions from Schrödinger, Bell, and others.
- First experiments and Nobel recognition.

Discussion Question

Why did Einstein describe entanglement as "spooky action at a distance"?

> communication and computing.[47]:218[48]:435Basically, knowing everything about the whole system doesn't mean you know everything about its parts.[49] For a pair of entangled particles, measuring one can tie strongly to what you get from the other. But it is not the everyday correlation, the potential for correlation that turns reality in the right setup.[50]:130 These links from entangled states can't be mimicked by classical odds.[51]:33


(Received March 25, 1935)

OCTOBER 15, 1935 PHYSICAL REVIEW VOLUME 48

Can Quantum-Mechanical Description of Physical Reality be Considered Complete?

N. Bohr, Institute for Theoretical Physics, University, Copenhagen

(Received July 13, 1935)

PHYSICAL REVIEW VOLUME 108, NUMBER 4 NOVEMBER 15, 1957

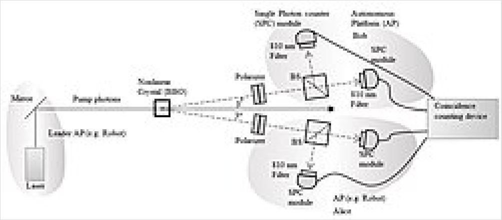
Discussion of Experimental Proof for the Paradox of Einstein, Rosen, and Podolsky

D. Bohm and Y. Aharonov

Technion, Haifa, Israel

(Received May 10, 1957)

Eprheaders



Quantum Entanglement Experiment via Spontaneous Parametric Down-Conversion (SPDC)

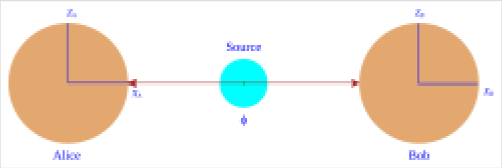
A subatomic particle splitting into an entangled duo. The split follows standard rules, so measuring one predicts the other (keeping totals like momentum or energy steady). A spin-zero particle breaks into two spin-1/2 ones. No orbital spin means total spin post-split is zero. Measure the first as spin up on an axis, the second's down on that axis. That's the anti-correlated singlet state. You might think hidden variables inside explain it, like one has "up," the other "down." Bell used the story of his pal Bertlmann, who always wore odd-colored socks: see one pink, know the other's not.[52] But to see quantum entanglement's true weirdness, you need various experiments, like spins on different axes, and compare those correlations.[53]:§18.8

Systems get entangled through different interactions. Check the methods section below for lab ways to make it happen. It breaks when particles decohere from environmental pokes, like measurements—the particles entangle with the surroundings, losing their own entanglement.[54]:369[55]

Math-wise, an entangled system's state can't break down into products of its parts' states, they are one unit. One part need the other.[56]:18–19[53]:§1.5 A combined system's state is a sum or superposition of local products; entangled if not reducible to one term.[47](p39)

Paradox

Main resource: EPR paradox]



EPR illustration

The singlet state is key to one take on the EPR paradox. In David Bohm's version, a source shoots particles opposite ways. Each pair's state is entangled.[57] Textbooks say measuring spin on one collapses the pair's wave function, giving each a definite spin (up or down) on that axis. It's random, 50-50. But same-axis measures are always opposite. So one measurement's random result seems sent to the other to match.[53]:§18.8[12]:447–448

You can set distances and timings so the measurements are spacelike, any cause linking them would beat light speed. Relativity says no info travels that fast. You can't even say which happened first; frames differ on order for spacelike events x1 and x2. So correlations aren't one determining the other, observers randomness incomplete.

But local hidden variable ideas flop with different-axis spins. Stats-wise, many pairs would meet Bell's inequality if local realism held. Experiments say no.[6][60][61][62] And in moving frames where one measure comes before the other, correlations hold.[63][43]:321–324

The big problem with different-axis spins: they can't have set values simultaneously, they're incompatible, limited by uncertainty. Unlike classical, where you measure anything together precisely. Math shows compatible measures can't violate Bell,[64] so entanglement's purely quantum.

Key Concepts

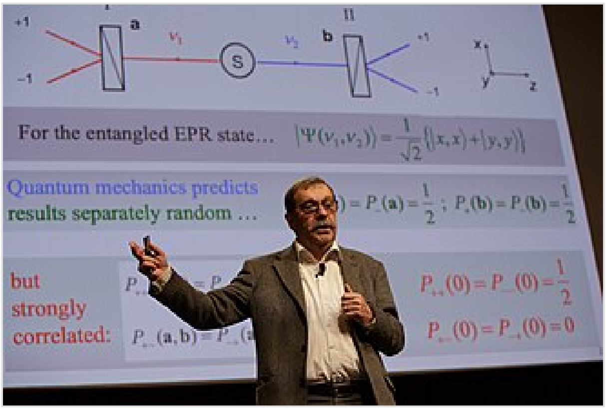
Singlet state and correlations.

- EPR paradox resolution via Bell inequalities.
- Incompatibility with local hidden variables.

Discussion Question

How do Bell inequalities demonstrate the non-classical nature of entanglement?

Module 4: Nonlocality, Resources, and Mathematical Details



Alain Aspect explaining his experiment

Nonlocality and entanglement

Alain Aspect's pioneering experiments in the early 1980s marked a turning point, providing strong evidence for quantum nonlocality. Using entangled photon pairs from calcium atomic cascades and rapidly switching polarizers (via acousto-optic modulators), his setup ensured measurement settings were chosen in a space-like separated manner, closing the locality loophole and demonstrating clear violations of Bell inequalities.

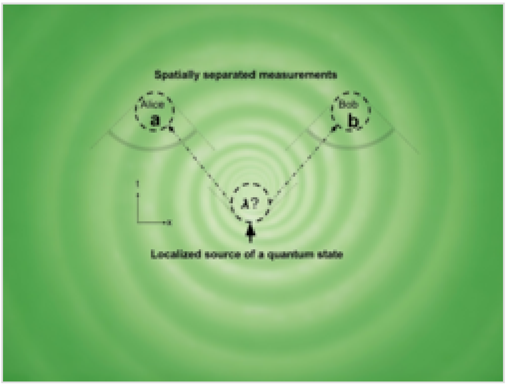
You need entanglement to break a Bell inequality. But just having it is not enough,[65] like Bell pointed out in '64.[33] Look at Werner states for pairs: some show entanglement but fit local hidden models, no Bell break.[66] Same for bigger groups.[67]

Breaking Bell inequalities gets called quantum nonlocality. The term stirs debate. Some say it hints wrongly at superluminal physical signals.[68] Failing local hidden models does not mean quantum's truly nonlocal.[69][70][71] But "nonlocality" stuck around anyway.[68] Sometimes "nonlocality" means other

Mathematical details

The following subsections use the formalism and theoretical framework developed in the articles bra–ket notation and mathematical formulation of quantum mechanics.

Pure states



Spacetime diagram for Bell's local hidden variable proof. Dash lines show relativistically valid region.

Consider two arbitrary quantum systems *A* and *B*, with respective Hilbert spaces *HA* and *HB*. The Hilbert space of the composite system is the tensor product

$$H_A \otimes H_B.$$

If the first system is in state $|\psi\rangle_A$ and the second in state $|\phi\rangle_B$, the state of the composite system is

$$|\psi\rangle_A \otimes |\phi\rangle_B.$$

States of the composite system that can be represented in this form are called separable states, or product states. However, not all states of the composite system are separable. Fix a basis $|i\rangle_A$ for *HA* and a basis $|j\rangle_B$ for *HB*. The most general state in *HA* ⊗ *HB* is of the form

$$|\psi\rangle_{AB} = \sum_{i,j} c_{ij} |i\rangle_A \otimes |j\rangle_B.$$

This state is separable if there exist vectors $[c^A_i]$, $[c^B_j]$ so that $c_{ij} = c^A_i c^B_j$, yielding $|\psi\rangle_A = \sum_i c^A_i |i\rangle_A$ and $|\phi\rangle_B = \sum_j c^B_j |j\rangle_B$. It is inseparable if for any vectors $[c^A_i]$, $[c^B_j]$ at least for one pair of coordinates c^A_i, c^B_j we have $c_{ij} \neq c^A_i c^B_j$. If a state is inseparable, it is called an 'entangled state'.[47]:218[53]:§1.5 For example, given two basis vectors $|0\rangle_A, |1\rangle_A$ of *HA* and two basis vectors $|0\rangle_B, |1\rangle_B$ of *HB*, the following is an entangled state:

$$\tfrac{1}{\sqrt{2}} \left(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B \right).$$

If the composite system is in this state, it is impossible to attribute to either system *A* or system *B* a definite pure state. Another way to say this is that while the von Neumann entropy of the whole state is zero (as it is for any pure state), the entropy of the subsystems is greater than zero. In this sense, the systems are "entangled". The above example is one of four Bell states, which are (maximally) entangled pure states (pure states of the $H_A \otimes H_B$ space, but which cannot be separated into pure states of each *HA* and *HB*).[53]:§18.6 Now suppose Alice is an observer for system *A*, and Bob is an observer for system *B*. If in the entangled state given above Alice makes a measurement in the $|0\rangle, |1\rangle$ eigenbasis of *A*, there remains true even if the systems A and B are spatially separated. This is the foundation of the EPR paradox. The outcome of Alice's measurement is random. Alice cannot decide which state to collapse the composite system into, and therefore cannot transmit information to Bob by acting on her system. Causality is thus preserved, in this particular scheme. For the general argument, see no-communication theorem.

Ensembles

As mentioned above, a state of a quantum system is given by a unit vector in a Hilbert space. More generally, if one has less information about the system, then it's called an 'ensemble', described by a density matrix, which is a positive-semidefinite matrix, or a trace class when the state space is infinite-dimensional, and which has trace 1. By the spectral theorem, such a matrix takes the general form:

$$\rho = \sum_i w_i |\alpha_i\rangle\langle\alpha_i|,$$

where the w_i are positive-valued probabilities (they sum up to 1), the vectors $|\alpha_i\rangle$ are unit vectors, and in the infinite-dimensional case, we would take the closure of such states in the trace norm. Interpret ρ as representing an ensemble where w_i is the proportion of the ensemble whose states are $|\alpha_i\rangle$. When a mixed state has rank 1, it therefore describes a 'pure ensemble'. When there is less than total information about the state of a quantum system a density matrices is used to represent the state.[54]:73–74[51]:13–15[53]:§22.2 Experimentally, a mixed ensemble might be realized as follows. Consider a "black box" apparatus that spits electrons towards an observer. The electrons' Hilbert spaces are identical. The apparatus might produce electrons that are all in the same state; in this case, the electrons received by the observer are then a pure ensemble. However, the apparatus could produce electrons in different states. For example, it could produce two populations of electrons: one with state $|\mathbf{z}+\rangle$ with spins aligned in the positive $\mathbf{z}$ direction, and the other with state $|\mathbf{y}-\rangle$ with spins aligned in the negative $\mathbf{y}$ direction. Generally, this is a mixed ensemble, as there can be any number of populations, each corresponding to a different state. Following the definition above, for a bipartite composite system, mixed states are just density matrices on $H_A \otimes H_B$. That is, it has the general form

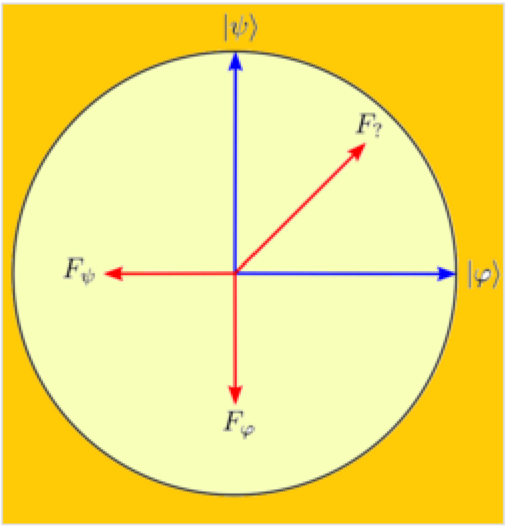
$$\rho = \sum_i w_i \left[\sum_j \bar{c}ij(|\alpha ij\rangle \otimes |\beta_{ij}\rangle)\right]\left[\sum_k c_{ik}(\langle\alpha_{ik}| \otimes \langle\beta_{ik}|)\right],$$

where the w_i are positively valued probabilities, $\sum_j |c_{ij}|^2 = 1$, and the vectors are unit vectors. This is self-adjoint and positive and has trace 1. Extending the definition of separability from the pure case, we say that a mixed state is separable if it can be written as

$$\rho = \sum_i w_i \rho_i^A \otimes \rho_i^B,$$

sums of pure ensembles and expanding, we may assume without loss of generality that ρ_i and ρ_i are themselves pure ensembles. A state is then said to be entangled if it is not separable. In general, finding out whether or not a mixed state is entangled is considered difficult. The general bipartite case has been shown to be NP-hard.[74] For the 2×2 and 2×3 cases, a necessary and sufficient criterion for separability is given by the famous Positive Partial Transpose (PPT) condition.[75]

Reduced density matrices



Bloch sphere representation of optimal POVM and states for unambiguous quantum state discrimination (yellow)

The idea of a reduced density matrix was introduced by Paul Dirac in 1930. Consider as above systems *A* and *B* each with a Hilbert space *HA*, *HB*. Let the state of the composite system be

$$|\Psi\rangle \in H_A \otimes H_B.$$

As indicated above, in general there is no way to associate a pure state to the component system *A*. However, it still is possible to associate a density matrix. Let

$$\rho_T = |\Psi\rangle\langle\Psi|,$$

which is the projection operator onto this state. The state of *A* is the partial trace of *ρT* over the basis of system *B*:

$$\rho_A \stackrel{\text{def}}{=} \sum_j^{N_B} \left(I_A \otimes \langle j|_B\right)\left(|\Psi\rangle\langle\Psi|\right)\left(I_A \otimes |j\rangle_B\right) = \mathrm{Tr}_B\,\rho_T,$$

The sum occurs over $N_B := \dim(H_B)$ and I_A the identity operator in H_A. *ρA* is sometimes called the reduced density matrix of *ρ* on subsystem *A*. For example, the reduced density matrix of *A* for the entangled state

$$\frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B\right),$$

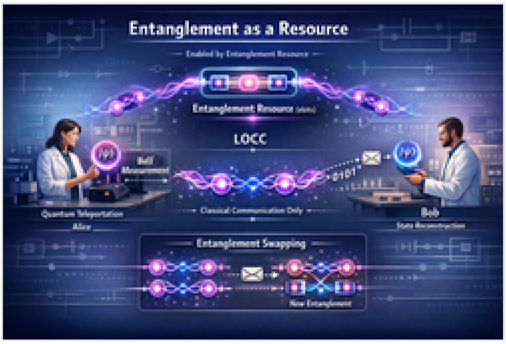
discussed above is[53]:§22.4

$$\rho_A = \frac{1}{2}\left(|0\rangle_A\langle 0|_A + |1\rangle_A\langle 1|_A\right).$$

This demonstrates that the reduced density matrix for an entangled pure ensemble is a mixed ensemble. In contrast, the density matrix of *A* for the pure product state $|\psi\rangle_A \otimes |\phi\rangle_B$ discussed above is

$$\rho_A = |\psi\rangle_A\langle\psi|_A,$$

Entanglement as a resource



Entanglement as a Resource

In quantum information theory, entangled states are considered a 'resource', i.e., something costly to produce and that allows implementing valuable transformations.

The setting in which this perspective is most evident is that of "distant labs", i.e., two quantum systems labelled "A" and "B" on each of which arbitrary quantum operations can be performed, but which do not interact with each other quantum mechanically. The only interaction allowed is the exchange of classical information, which combined with the most general local quantum operations gives rise to the class of operations called LOCC (local operations and classical communication). These operations do n|LOCCot allow the production of entangled states between systems A and B. But if A and B are provided with a supply of entangled states, then these, together with LOCC operations can enable a larger class of transformations.

If Alice and Bob share an entangled state, Alice can tell Bob over a telephone call how to reproduce a quantum state $|\Psi\rangle$ she has in her lab. Alice performs a joint measurement on $|\Psi\rangle$ together with her half of the entangled state and tells Bob the results. Using Alice's results Bob operates on his half of the entangled state to make it equal to $|\Psi\rangle$. Since Alice's measurement necessarily erases the quantum state of the system in her lab, the state $|\Psi\rangle$ is not copied, but transferred: it is said to be "teleported" to Bob's laboratory through this protocol.[1]:875[76]

Entanglement swapping is variant of teleportation that allows two parties that have never interacted to share an entangled state. The swapping protocol begins with two EPR sources. One source emits an entangled pair of particles A and B, while the other emits a second entangled pair of particles C and D. Particles B and C are subjected to a measurement in the basis of Bell states. The state of the remaining particles, A and D, collapses to a Bell state, leaving them entangled despite never having interacted with each other.[1][77]

An interaction between a qubit of A and a qubit of B can be realized by first teleporting A's qubit to B, then letting it interact with B's qubit (which is now a LOCC operation, since both qubits are in B's lab) and then teleporting the qubit back to A. Two maximally entangled states of two qubits are used up in this process. So entangled states are a resource that enables the realization of quantum interactions (or of quantum channels) in a setting where only LOCC are available, but they are consumed in the process. There are other applications where entanglement can be seen as a resource, e.g., private communication or distinguishing quantum states.[1]

Key Concepts

- Quantum nonlocality and Bell inequalities.
- Pure and mixed states, density matrices.

How does entanglement enable quantum teleportation without violating relativity?

Module 5: Multipartite Entanglement, Measures, and Applications

Multipartite entanglement

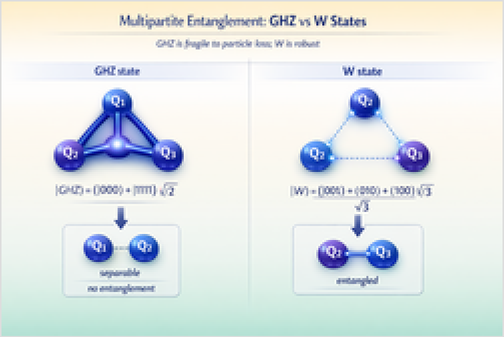
Main resource: Multipartite entanglement

Quantum states describing systems made of more than two pieces can also be entangled. An example for a three-qubit system is the Greenberger–Horne–Zeilinger (GHZ) state,

$$|\mathrm{GHZ}\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}.$$

Another three-qubit example is the W state:

$$|\mathrm{W}\rangle = \frac{|001\rangle + |010\rangle + |100\rangle}{\sqrt{3}}.$$



Multipartite Entanglement GHZ vs W states

Tracing out any one of the three qubits turns the GHZ state into a separable state, whereas the result of tracing over any of the three qubits in the W state is still entangled. This illustrates how multipartite entanglement is a more complicated topic than bipartite entanglement: systems composed of three or more parts can exhibit multiple qualitatively different types of entanglement.[48]:493–497 A single particle cannot be maximally entangled with more than a particle at a time, a property called monogamy.[78]

Classification of entanglement. Not all quantum states are equally valuable as a resource. One method to quantify this value is to use an entanglement measure that assigns a numerical value to each quantum state. However, it is often interesting to settle for a coarser way to compare quantum states. This gives rise to different classification schemes. Most entanglement classes are defined based on whether states can be converted to other states using LOCC or a subclass of these operations. The smaller the set of allowed operations, the finer the classification. Important examples are:

If two states can be transformed into each other by a local unitary operation, they are said to be in the same *LU class*. This is the finest of the usually considered classes. Two states in the same LU class have the same value for entanglement measures and the same value as a resource in the distant-labs setting. There is an infinite number of different LU classes (even in the simplest case of two qubits in a pure state).[79][80]

If two states can be transformed into each other by local operations including measurements with probability larger than 0, they are said to be in the same *SLOCC class* ("stochastic LOCC"). Qualitatively, two states ρ_1 and ρ_2 in the same SLOCC class are equally powerful, since one can transform each into the other, but since the transformations $\rho_1 \to \rho_2$ and $\rho_2 \to \rho_1$ may succeed with different probability, Instead of considering transformations of single copies of a state (like $\rho_1 \to \rho_2$) one can define classes based on the possibility of multi-copy transformations. E.g., there are examples when $\rho_1 \to \rho_2$ is impossible by LOCC, but $\rho_1 \otimes \rho_1 \to \rho_2$ is possible. A very important (and very coarse) classification is based on the property whether it is possible to transform an arbitrarily large number of copies of a state ρ into at least one pure entangled state. States that have this property are called distillable. These states are the most useful quantum states since, given enough of them, they can be transformed (with local operations) into any entangled state and hence allow for all possible uses. It came initially as a surprise that not all entangled states are distillable; those that are not are called "bound entangled".[83][1]

A different entanglement classification is based on what the quantum correlations present in a state allow A and B to do: one distinguishes three subsets of entangled states:

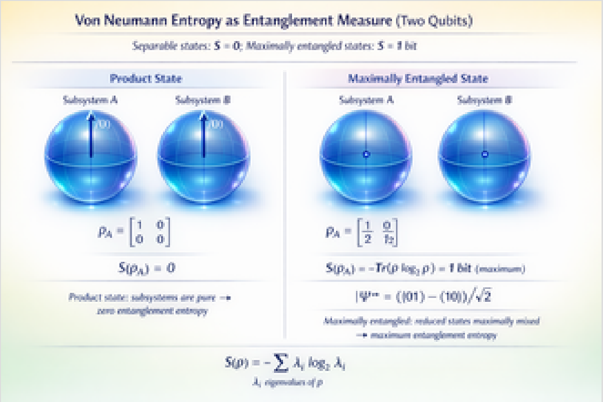
(1) the *non-local states*, which produce correlations that cannot be explained by a local hidden variable model and thus violate a Bell inequality,

(2) the *steerable states* that contain sufficient correlations for A to modify ("steer") by local measurements the conditional reduced state of B in such a way, that A can prove to B that the state they possess is indeed entangled, and finally

(3) those entangled states that are neither non-local nor steerable. All three sets are non-empty.[84]

Entropy

In this section, the entropy of a mixed state is discussed as well as how it can be viewed as a measure of quantum entanglement.



Von Neumann Entropie as Entanglement Measure

Definition In classical H, the Shannon entropy, is associated to a probability distribution, $p_1, \cdots, p_n$, in the following way:[85]

$$H(p_1, \cdots, p_n) = -\sum_i p_i \log_2 p_i .$$

Since a mixed state ρ is a probability distribution over an ensemble, this leads naturally to the definition of the von Neumann entropy:[54]:264

$$S(\rho) = -\mathrm{Tr}\left(\rho \log_2 \rho\right),$$

which can be expressed in terms of the eigenvalues of ρ:

$$S(\rho) = -\sum_i \lambda_i \log_2 \lambda_i .$$

the convention $0 \log 0 = 0$ is adopted. When a pair of particles is described by the spin singlet state discussed above, the von Neumann entropy of either particle is $\log(2)$, which can be shown to be the maximum entropy for 2×2 mixed states.[51]:15

As a measure of entanglement Entropy provides one tool that can be used to quantify entanglement, although other entanglement measures exist.[86][87]

If the overall system is pure, the entropy of one subsystem can be used to measure its degree of entanglement with the other subsystems. For bipartite pure states, the von Neumann entropy of reduced states is the unique measure of entanglement in the sense that it is the only function on the family of states that satisfies certain axioms required of an entanglement measure.[88]

It is a classical result that the Shannon entropy achieves its maximum at, and only at, the uniform probability distribution $1/n, \ldots, 1/n$.

Therefore, a bipartite pure state $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$ is said to be a *maximally entangled state* if the reduced state of each subsystem of ρ is the diagonal matrix:[89]

$$\begin{bmatrix} \frac{1}{n} & & \\ & \ddots & \\ & & \frac{1}{n} \end{bmatrix}.$$

For mixed states, the reduced von Neumann entropy is not the only reasonable entanglement measure.[48]:471 Rényi entropy also can be used as a measure of entanglement.[48]:447,480[90]

Entanglement measures

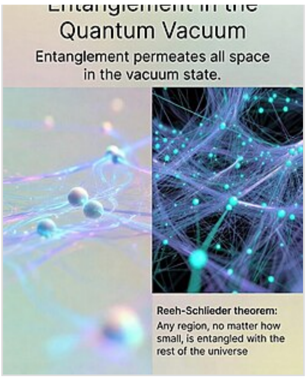
Entanglement measures quantify the amount of entanglement in a (often viewed as a bipartite) quantum state. As aforementioned, entanglement entropy is the standard measure of entanglement for pure states (but no longer a measure of entanglement for mixed states). For mixed states, there are some entanglement measures in the literature[86] and no single one is standard.

Entanglement cost

- Distillable entanglement
- Entanglement of formation
- Concurrence
- Relative entropy of entanglement
- Squashed entanglement
- Logarithmic negativity

Most (but not all) of these entanglement measures reduce for pure states to entanglement entropy, and are difficult (NP-hard) to compute for mixed states as the dimension of the entangled system grows.[91]

vacuum.[92]



Reeh-Schlieder Theorem

Applications

Entanglement has many applications in quantum information theory. With the aid of entanglement, otherwise impossible tasks may be achieved. Among the best-known applications of entanglement are superdense coding and quantum teleportation.[41] Most researchers believe that entanglement is necessary to realize quantum computing (although this is disputed by some).[93]

Entanglement is used in some protocols of quantum cryptography,[38] but to prove the security of quantum key distribution (QKD) under standard assumptions does not require entanglement.[94] However, the *device independent* security of QKD is shown exploiting entanglement between the communication partners.[95]

In August 2014, Brazilian researcher Gabriela Barreto Lemos and team were able to "take pictures" of objects using photons that had not interacted with the subjects, but were entangled with photons that did interact with such objects.[96] This idea has been adapted to make infrared images using standard cameras insensitive to infrared.[97]

Entangled states

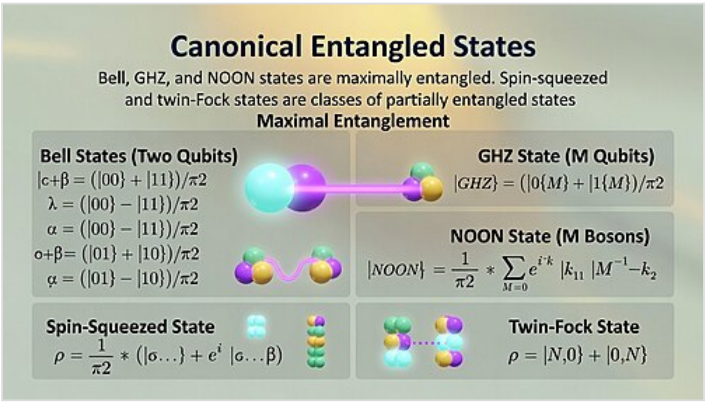
There are several canonical entangled states that appear often in theory and experiments.

For two qubits, the Bell states are:

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B \pm |1\rangle_A \otimes |1\rangle_B)$$

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B \pm |1\rangle_A \otimes |0\rangle_B)$$

These four pure states are all maximally entangled and form an orthonormal basis of the Hilbert space of the two qubits.



Canonical Entangled States in Quantum Theory

For $M > 2$ qubits, the GHZ state is:

$$|\mathrm{GHZ}\rangle = \frac{|0\rangle^{\otimes M} + |1\rangle^{\otimes M}}{\sqrt{2}},$$

measurements.[100][101]

For two bosonic modes, a NOON state is:

$$|\psi_{\text{NOON}}\rangle = \frac{|N\rangle_a|0\rangle_b + |0\rangle_a|N\rangle_b}{\sqrt{2}}.$$

Finally, twin Fock states for bosonic modes can be used to reach the Heisenberg limit|Heisenberg limit.[102]

Bell, GHZ, and NOON states are maximally entangled, while spin squeezed and twin Fock states are only partially entangled.[103][104]

Methods of creating entanglement

- Entanglement usually comes from direct particle interactions. Common methods include:
- Spontaneous parametric down-conversion (photon pairs entangled in polarization).[1]
- Fiber couplers to mix photons or bi-exciton decay in quantum dots.[105]
- The Hong–Ou–Mandel effect.[106]
- Particle-antiparticle partial wavefunction overlap (Hardy's interferometer).
- Systems that never interact directly can be entangled via swapping or wavefunction overlap.[107]

Testing a system for entanglement

A density matrix ρ is called separable if it can be written as a convex sum of product states:

$$\rho = \sum_j p_j \rho_j^{(A)} \otimes \rho_j^{(B)}, \quad 0 \leq p_j \leq 1.$$

By definition, a state is entangled if it is not separable. For 2-qubit and qubit-qutrit systems, the Peres–Horodecki criterion is necessary and sufficient; for general cases, it is merely necessary. Other criteria include the range criterion, reduction criterion, and uncertainty relation-based tests.

Continuous variable systems use Simon's condition for 1⊕1-mode Gaussian states, generalized for higher modes. Entropic measures also provide entanglement criteria.

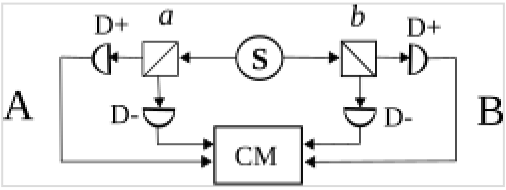
In quantum gravity

Entanglement may explain the "problem of time": in quantum mechanics, time is a fixed backdrop, while in general relativity, it is dynamic. The Wheeler–DeWitt equation suggests a static universe.

Experiments demonstrating and using entanglement

Bell tests

Main resource: Bell test

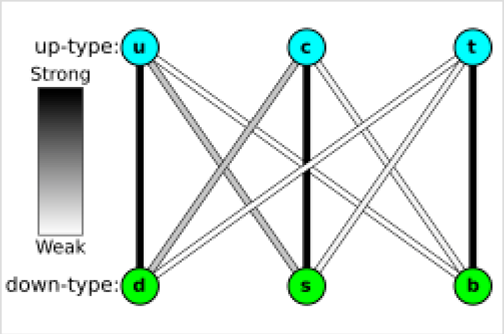


Two channel bell test

A Bell test, or Bell inequality test or experiment, is a lab setup to pit quantum mechanics against local hidden variables. They check Bell's theorem predictions. So far, every one shows local hidden variables don't match reality. Labs run many to fix design flaws that might skew earlier results. closing loopholes. Early ones couldn't rule out sneaky signals from one site to the other.[9] But "loophole-free" tests space sites so light-speed comms take longer, one case, 10,000 times longer, than measurement time.[8][7][15][36]

In 2017, Yin and team set a 1,203 km record for entanglement, showing two-photon survival and Bell violation (CHSH 2.37±0.09) under strict locality, from Micius satellite to bases in Yunnan and Qinghai, upping efficiency ten times over fiber.[108][109]

Entanglement of top quarks



Quark weak interactions

In 2023, the LHC used techniques from quantum tomography to measure entanglement at the highest energy so far.[110][111][112] This work is based on theoretical proposals from 2021.[113]

The experiment was carried out by the ATLAS detector, which measured the spin of top-quark pair production. The effect was observed with a significance of more than 5σ. The top quark is the heaviest known particle and therefore has a very short lifetime, approximately 10^{-25} s, making it the only quark that decays before undergoing hadronization ($\sim 10^{-23}$ s) and spin decorrelation ($\sim 10^{-21}$ s). As a result, the spin information is transferred without significant loss to the leptonic decay products captured by the detector.[114]

The spin polarization and correlation of the particles were measured and tested for entanglement using concurrence as well as the Peres–Horodecki criterion. The effect has also been confirmed independently by the CMS detector.[115][116]

- Quantum
- Quantum A Matter Of Size
- Quantum A Spooky Action at a Distance
- Quantum: A Walk Through the Universe
- Number of independent spatial modes in a spherical volume
- Quantum Computing Algorithms in the NISQ Era
- Quantum Formulas Collection
- Quantum Matter Elements and Particles
- Quantum mechanics
- Quantum mechanics/Timeline
- Quantum mechanics measurements
- Quantum Noisy Qubits
- Quantum optics beam splitter experiments
- Quantum: The Secret of Cohesion: How Waves Hold Matter Together
- Quantum Ultra fast lasers
- Template:Quantum optics operators
- Physical Sciences

- Concurrence
- CNOT gate
- Einstein's thought experiments
- Entanglement witness
- ER = EPR
- Multipartite entanglement
- Normally distributed and uncorrelated does not imply independent
- Pauli exclusion principle
- Quantum coherence
- Quantum discord
- Quantum network
- Quantum phase transition
- Quantum pseudo-telepathy
- Retrocausality
- Squashed entanglement
- Stern–Gerlach experiment
- Ward's probability amplitude

Physics portal

Key Concepts

Multipartite states like GHZ and W states. Entanglement measures and classification (LU, SLOCC). Applications in computing, cryptography, and experiments.

Discussion Question

What are the practical implications of using entanglement in quantum computing?

Course Quiz

Test your knowledge with this short quiz.
Answers are provided below.

What term did Einstein use to describe quantum entanglement?
a) Local realism
b) Spooky action at a distance
c) Hidden variables
d) Wave function collapse

Which experiment first demonstrated loophole-free Bell inequality violation using electron spins?
a) Aspect's experiment

True or False: Quantum entanglement can be used to send information faster than light.

What is the von Neumann entropy used for in the context of entanglement?
a) Measuring classical correlations
b) Quantifying entanglement in pure states
c) Calculating particle spin
d) Determining wave function probability
Name one application of quantum entanglement in quantum information theory.

Further Reading

- Albert, David Z.; Galchen, Rivka (2009). "Was Einstein Wrong?: A Quantum Threat to Special Relativity". *Scientific American* **300** (3): 32–39. doi:10.1038/scientificamerican0309-32. PMID 19253771. https://www.scientificamerican.com/article/was-einstein-wrong-about-relativity/.
- Cramer, J. G. (2015). *The Quantum Handshake: Entanglement, Nonlocality and Transactions*. Springer Verlag. ISBN 978-3-319-24642-0.
- Duarte, F. J. (2019). *Fundamentals of Quantum Entanglement*. Bristol, United Kingdom: Institute of Physics. ISBN 978-0-7503-2226-3.
- "Generalized concurrence measure for faithful quantification of multiparticle pure state entanglement using Lagrange's identity and wedge product". *Quantum Information Processing* **16** (5). 2017. doi:10.1007/s11128-017-1568-0.
- "Generalized entanglement measure for continuous-variable systems". *Physical Review A* **105** (5). 2022. doi:10.1103/PhysRevA.105.052441.
- Jaeger, G. (2009). *Entanglement, Information, and the Interpretation of Quantum Mechanics*. Heildelberg, Germany: Springer. ISBN 978-3-540-92127-1.
- Steward, E. G. (2008). *Quantum Mechanics: Its Early Development and the Road to Entanglement*. Imperial College Press. ISBN 978-1-86094-978-4.
- Wilde, Mark M. (2017). *Quantum Information Theory* (2nd ed.). Cambridge University Press. doi:10.1017/9781316809976. ISBN 978-1-316-80997-6.

External links

- Explanatory video by *Scientific American* magazine (https://www.youtube.com/watch?v=xM3GOXaci7w)
- Entanglement experiment with photon pairs – interactive (https://web.archive.org/web/20121025073450/http://www.didaktik.physik.uni-erlangen.de/quantumlab/english/index.html)
- Audio – Cain/Gay (2009) Astronomy Cast (http://www.astronomycast.com/physics/ep-140-entanglement/) Entanglement
- "Spooky Actions at a Distance?": Oppenheimer Lecture, Prof. David Mermin (Cornell University) Univ. California, Berkeley, 2008. (https://www.youtube.com/watch?v=ta09WXiUqcQ) Non-mathematical popular lecture on YouTube, posted Mar 2008

Search Wikiquote for quotations related to: ***Quantum A Spooky Action at a Distance***

Quiz Answers

b) Spooky action at a distance
b) Hensen et al. (2015)
False
b) Quantifying entanglement in pure states
Quantum teleportation, superdense coding, or quantum cryptography (any one is acceptable)

Quantum mechanics	**Search for *Quantum A Spooky Action at a Distance* on Wikipedia.**
Background	Introduction · History (Timeline) · Classical mechanics · Old quantum theory · Glossary
Fundamentals	Born rule · Bra–ket notation · Complementarity · Density matrix · Energy level (Ground state · Excited state · Degenerate levels · Zero-point energy) · Entanglement · Hamiltonian · Interference · Decoherence · Measurement · Nonlocality · Quantum state (quantum jump) · Superposition · Tunnelling · Scattering theory · Symmetry in quantum mechanics · Uncertainty · Wave function (Collapse · Wave–particle duality)
Formulations	Formulations · Heisenberg · Interaction · Matrix mechanics · Schrödinger · Path integral formulation · Phase space
Equations	Klein–Gordon · Dirac · Weyl · Majorana · Rarita–Schwinger · Pauli · Rydberg · Schrödinger
Interpretations	Bayesian · Consciousness causes collapse · Consistent histories · Copenhagen · de Broglie–Bohm · Ensemble · Hidden-variable (Local (Superdeterminism)) · Many-worlds · Objective collapse · Quantum logic · Relational · Transactional
Experiments	Bell test · Davisson–Germer · Delayed-choice quantum eraser · Double-slit · Franck–Hertz · Mach–Zehnder interferometer · Elitzur–Vaidman · Popper · Quantum eraser · Stern–Gerlach · Wheeler's delayed choice
Science	Quantum biology · Quantum chemistry · Quantum chaos · Quantum cosmology · Quantum differential calculus · Quantum dynamics · Quantum geometry · Quantum measurement problem · Quantum mind · Quantum stochastic calculus · Quantum spacetime
Technology	Quantum algorithms · Quantum amplifier · Quantum bus · Quantum cellular automata (Quantum finite automata) · Quantum channel · Quantum circuit · Quantum complexity theory · Quantum computing (Timeline) · Quantum cryptography · Quantum electronics · Quantum error correction · Quantum imaging · Quantum image processing · Quantum information · Quantum key distribution · Quantum logic · Quantum logic gates · Quantum machine · Quantum machine learning · Quantum metamaterial · Quantum metrology · Quantum network · Quantum neural network · Quantum optics ·

	Quantum field theory (History) · Quantum gravity · Relativistic quantum mechanics
Related	Schrödinger's cat (in popular culture) · Wigner's friend · EPR paradox · Quantum mysticism
Category:Quantum mechanics	

Horodecki, P., Horodecki, M., & Horodecki, K. (2009). Quantum entanglement. Reviews of Modern Physics, 81(2), 865–942. https://doi.org/10.1103/RevModPhys.81.865

2. Einstein, Albert; Podolsky, Boris; Rosen, Nathan (1935). "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?". *Phys. Rev.* **47** (10): 777–780. doi:10.1103/PhysRev.47.777.
3. Schrödinger, Erwin (1935). "Discussion of probability relations between separated systems". *Mathematical Proceedings of the Cambridge Philosophical Society* **31** (4): 555–563. doi:10.1017/S0305004100013554.
4. Schrödinger, Erwin (1936). "Probability relations between separated systems". *Mathematical Proceedings of the Cambridge Philosophical Society* **32** (3): 446–452. doi:10.1017/S0305004100019137.

depicts the Einstein camp in this debate in his article entitled "Bertlmann's socks and the nature of reality", p. 143 of *Speakable and unspeakable in quantum mechanics*: "For EPR that would be an unthinkable 'spooky action at a distance'. To avoid such action at a distance they have to attribute, to the space-time regions in question, real properties in advance of observation, correlated properties, which predetermine the outcomes of these particular observations. Since these real properties, fixed in advance of observation, are not contained in quantum formalism, that formalism for EPR is incomplete. It may be correct, as far as it goes, but the usual quantum formalism cannot be the whole story." And again on p. 144 Bell says: "Einstein had no difficulty accepting that affairs in different places could be correlated. What he could not accept was that an intervention at one place could influence, immediately, affairs at the other." Downloaded 5 July 2011 from Bell, J. S. (1987). *Speakable and Unspeakable in Quantum Mechanics*. CERN. ISBN 0-521-33495-0. http://philosophyfaculty.ucsd.edu/faculty/wuthrich/GSSPP09/Files/BellJohnannsSocks.pdf.

6. Freedman, Stuart J.; Clauser, John F. (1972). "Experimental Test of Local Hidden-Variable Theories". *Physical Review Letters* **28** (14): 938–941. doi:10.1103/PhysRevLett.28.938. https://escholarship.org/uc/item/2f18n5nk.
7. Yin, Juan; Cao, Yuan; Yong, Hai-Lin; Ren, Ji-Gang *et al*. (2013). "Bounding the speed of 'spooky action at a distance". *Physical Review Letters* **110** (26). doi:10.1103/PhysRevLett.110.260407. PMID 23848853.
8. Matson, John (13 August 2012). "Quantum teleportation achieved over record distances". *Nature News*. doi:10.1038/nature.2012.11163.
9. Francis, Matthew (30 October 2012). "Quantum entanglement shows that reality can't be local". *Ars Technica*. Retrieved 22 August 2023.
10. Penrose, Roger (2004). *The road to reality: a complete guide to the laws of the universe*. London: Jonathan Cape. p. 603. ISBN 978-0-224-04447-9.
11. Siegel, Ethan. "No, We Still Can't Use Quantum Entanglement To Communicate Faster Than Light". *Starts with a Bang*. Forbes. Retrieved 6 January 2023.

ed.). Cambridge University Press. ISBN 978-1-107-18963-8.

13. Kocher, C. A.; Commins, E. D. (1967). "Polarization Correlation of Photons Emitted in an Atomic Cascade". *Physical Review Letters* **18** (15): 575–577. doi:10.1103/PhysRevLett.18.575. http://www.escholarship.org/uc/item/1kb7660q.
14. Kocher, Carl Alvin (1 May 1967). *Polarization Correlation of Photons Emitted in an Atomic Cascade* (PhD thesis). University of California.
15. Hensen, B. (21 October 2015). "Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres". *Nature* **526** (7575): 682–686. doi:10.1038/nature15759. PMID 26503041. See also free online access version (http://www.nature.com/articles/nature15759.epdf?referrer_access_token=1QB20mTNTZW60nEXil0D79RgN0jAjWel9jnR3ZoTv0Pfu6MWINxm4Io03p2jIRZ8qX_3I3N0Kr-AIltuikCZOJrG8QbdRRghlecFwmixlbQpWuw1dtaib4Le5DQOG3u_aXHU85x1JEhOcQTa1sHi0yvW23bblxmEQZAmHL4G0gIVusG_6JWorroY5BprgbTI4FiaE8WltEgMoUMZfZBkEfbMcFDp5iR112TFx_x3ZRj88Wa23E2moEvTfKjtlued0&tracking_referrer=www.nytimes.com).

Is Real". *The New York Times*. Retrieved 21 October 2015.

17. Boerkamp, Martijn (11 October 2023). "Quantum entanglement observed in top quarks". *W:Physics World*.
18. Holland, Connor M.; Lu, Yukai; Cheuk, Lawrence W. (8 December 2023). "On-demand entanglement of molecules in a reconfigurable optical tweezer array". *Science* **382** (6675): 1143–1147. doi:10.1126/science.adf4272. ISSN 0036-8075. PMID 38060644. https://www.science.org/doi/10.1126/science.adf4272.
19. Lee, K. C.; Sprague, M. R.; Sussman, B. J.; Nunn, J. *et al*. (2 December 2011). "Entangling macroscopic diamonds at room temperature". *Science* **334** (6060): 1253–1256. doi:10.1126/science.1211914. PMID 22144620. http://scholarbank.nus.edu.sg/handle/10635/112433.

1935: Einstein's Early Worries About The Quantum Mechanics of Composite Systems". In Miller, A. I.. *Sixty-Two Years of Uncertainty*. New York: Plenum Press. pp. 61–111. http://www.ub.edu/hcub/hfq/sites/default/files/Howard1990-1.pdf.

21. Weyl, Hermann (1931). *Gruppentheorie und Quantenmechanik* (2nd ed.). pp. 92–93.
22. Heathcote, Adrian (2021). "Multiplicity and indiscernability". *Synthese* **198** (9): 8779–8808. doi:10.1007/s11229-020-02600-8. "For Weyl clearly anticipated entanglement by noting that the pure state of a coupled system need not be determined by the states of the composites [...] Weyl deserves far more credit than he has received for laying out the basis for entanglement — more than six years before Schrödinger coined the term."
23. Christandl, Matthias (2006). *The Structure of Bipartite Quantum States – Insights from Group Theory and Cryptography* (PhD thesis). University of Cambridge. pp. vi, iv. arXiv:quant-ph/0604183. Bibcode:2006PhDT.......289C.

Elektrons' and its Influence on Grete Hermann". *Grete Hermann – Between Physics and Philosophy*. Studies in History and Philosophy of Science. **42**. Springer. p. 76. doi:10.1007/978-94-024-0970-3_5. ISBN 978-94-024-0968-0.

25. Kumar, Manjit (2010). *Quantum: Einstein, Bohr, and the Great Debate about the Nature of Reality*. W. W. Norton & Company. p. 313. ISBN 978-0-393-07829-9.
26. Schroeder, Daniel V. (1 November 2017). "Entanglement isn't just for spin". *American Journal of Physics* **85** (11): 812–820. doi:10.1119/1.5003808. ISSN 0002-9505. https://pubs.aip.org/ajp/article/85/11/812/1057936/Entanglement-isn-t-just-for-spin.
27. "Introduction". *Philosophy of Quantum Information and Entanglement*. Cambridge University Press. 2010. p. xv. ISBN 978-0-511-67655-0.

between Albert Einstein and Max and Hedwig Born from 1916 to 1955, Walker, New York, 1971. Cited in Hobson, M. P. (1998). "Quantum Entanglement and Communication Complexity". *SIAM J. Comput.* **30** (6): 1829–1841.)

29. Mermin, N. David (1985). "Is the Moon There When Nobody Looks? Reality and the Quantum Theory". *Physics Today* **38** (4): 38–47. doi:10.1063/1.880968. https://archive.org/details/mermin_moon.
30. Wheeler, J. A. (1946). "Polyelectrons". *Annals of the New York Academy of Sciences* **48** (3): 219–238. doi:10.1111/j.1749-6632.1946.tb31764.x.
31. Wu, C. S.; Shaknov, I. (1950). "The Angular Correlation of Scattered Annihilation Radiation". *W:Physical Review* **77** (1): 136. doi:10.1103/PhysRev.77.136.
32. Duarte, F. J. (2012). "The origin of quantum entanglement experiments based on polarization measurements". *W:European Physical Journal H* **37** (2): 311–318. doi:10.1140/epjh/e2012-20047-y.

195–200. doi:10.1103/PhysicsPhysiqueFizika.1.195.

34. Mermin, N. David (1981). "Quantum Mysteries for Anyone". *The Journal of Philosophy* **78** (7): 397–408. doi:10.2307/2026482. ISSN 0022-362X.
35. Aspect, Alain; Grangier, Philippe; Roger, Gérard (1982). "Experimental Realization of Einstein–Podolsky–Rosen–Bohm Gedankenexperiment: A New Violation of Bell's Inequalities". *Physical Review Letters* **49** (2): 91–94. doi:10.1103/PhysRevLett.49.91.
36. Hanson, Ronald (2015). "Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres". *Nature* **526** (7575): 682–686. doi:10.1038/nature15759. PMID 26503041.
37. Aspect, Alain (16 December 2015). "Closing the Door on Einstein and Bohr's Quantum Debate". *Physics* **8**. doi:10.1103/Physics.8.123.
38. Ekert, Artur K. (1991). "Quantum cryptography based on Bell's theorem". *Physical Review Letters* **67** (6): 661–663. doi:10.1103/PhysRevLett.67.661. PMID 10044956.

Wootters, William K. (29 March 1993). "Teleporting an Unknown Quantum State via Dual Classical and Einstein–Podolsky–Rosen Channels". *W:Physical Review Letters* **70** (13): 1895–1899. doi:10.1103/PhysRevLett.70.1895. PMID 10053414.

40. Lindley, David (8 January 2010). "Landmarks: Teleportation is not Science Fiction". *Physics (Physical Review Focus)* **25**. https://physics.aps.org/story/v25/st1.
41. Bouwmeester, Dik; Pan, Jian-Wei; Mattle, Klaus; Eibl, Manfred; Weinfurter, Harald; Zeilinger, Anton (1 December 1997). "Experimental quantum teleportation". *Nature* **390** (6660): 575–579. doi:10.1038/37539.
42. Boschi, D.; Branca, S.; De Martini, F.; Hardy, L.; Popescu, S. (9 February 1998). "Experimental Realization of Teleporting an Unknown Pure Quantum State via Dual Classical and Einstein–Podolsky–Rosen Channels". *Physical Review Letters* **80** (6): 1121–1125. doi:10.1103/PhysRevLett.80.1121.

reborn (1. Vintage Book ed.). New York, NY: Vintage Books. ISBN 978-1-4000-9526-1.

44. Jennewein, T.; Simon, C.; Weihs, G.; Weinfurter, H.; Zeilinger, A. (2000). "Quantum Cryptography with Entangled Photons". *Physical Review Letters* **84** (20): 4729–4732. doi:10.1103/PhysRevLett.84.4729. PMID 10990782.
45. Del Santo, F; Schwarzhans, E. (2022). ""Philosophysics" at the University of Vienna: The (Pre-) History of Foundations of Quantum Physics in the Viennese Cultural Context". *Physics in Perspective* **24** (2–3): 125–153. doi:10.1007/s00016-022-00290-y. PMID 36437910. PMC 9678993. //www.ncbi.nlm.nih.gov/pmc/articles/PMC9678993/.
46. "The Nobel Prize in Physics 2022" (Press release). The Royal Swedish Academy of Sciences. 4 October 2022. Retrieved 5 October 2022.
47. Rieffel, Eleanor; Polak, Wolfgang (2011). *Quantum Computing: A Gentle Introduction*. Scientific and engineering computation. Cambridge, Mass: MIT Press. ISBN 978-0-262-01506-6.

Introduction to Quantum Entanglement (2nd ed.). Cambridge University Press. ISBN 978-1-107-02625-4.

49. Rau, Jochen (2021). *Quantum Theory: An Information Processing Approach*. Oxford University Press. ISBN 978-0-19-289630-8.
50. Fuchs, Christopher A. (6 January 2011). *Coming of Age with Quantum Information*. Cambridge University Press. ISBN 978-0-521-19926-1.
51. Holevo, Alexander S. (2001). *Statistical Structure of Quantum Theory*. Lecture Notes in Physics. Monographs. Springer. ISBN 3-540-42082-7.
52. Bell, J. (1981). "Bertlmann's Socks and the Nature of Reality". *Journal de Physique Colloques* **42 (C2)**: 41–62. doi:10.1051/jphyscol:1981202. https://hal.science/jpa-00220688v1.
53. Zwiebach, Barton (2022). *Mastering Quantum Mechanics: Essentials, Theory, and Applications*. MIT Press. ISBN 978-0-262-04613-8.
54. Peres, Asher (1993). *Quantum Theory: Concepts and Methods*. Kluwer. ISBN 0-7923-2549-4.

57. doi:10.1016/j.physrep.2019.10.001.

56. Mermin, N. David (2007). *Quantum Computer Science: An Introduction*. Cambridge University Press. ISBN 978-0-521-87658-2.
57. Bohm, David (1989). *Quantum Theory* (reprint ed.). Dover. pp. 611–622. ISBN 0-486-65969-0.
58. Peres, Asher (2000-01-18). "Classical interventions in quantum systems. II. Relativistic invariance". *Physical Review A* **61** (2). doi:10.1103/PhysRevA.61.022117.
59. Gibney, Elizabeth (2017). "Cosmic Test Bolsters Einstein's "Spooky Action at a Distance"". *Scientific American*. https://www.scientificamerican.com/article/cosmic-test-bolsters-einsteins-ldquo-spooky-action-at-a-distance-rdquo/.
60. Dehlinger, Dietrich; Mitchell, M. W. (2002). "Entangled photons, nonlocality, and Bell inequalities in the undergraduate laboratory". *American Journal of Physics* **70** (9): 903–910. doi:10.1119/1.1498860.
61. BIG Bell Test Collaboration (May 2018). "Challenging local realism with human choices". *Nature* **557** (7704): 212–216. doi:10.1038/s41586-018-0085-3. PMID 29743691.

from High-Redshift Quasars". *Physical Review Letters* **121** (8). doi:10.1103/PhysRevLett.121.080403. PMID 30192604.

63. Zbinden, H. *et al*. (2001). "Experimental test of nonlocal quantum correlations in relativistic configurations". *Physical Review A* **63** (2). doi:10.1103/PhysRevA.63.022111. http://archive-ouverte.unige.ch/unige:37034.
64. Cirel'son, B. S. (1980). "Quantum generalizations of Bell's inequality". *Letters in Mathematical Physics* **4** (2): 93–100. doi:10.1007/BF00417500.
65. Brunner, Nicolas; Cavalcanti, Daniel; Pironio, Stefano; Scarani, Valerio; Wehner, Stephanie (2014). "Bell nonlocality". *Reviews of Modern Physics* **86** (2): 419–478. doi:10.1103/RevModPhys.86.419.
66. Werner, R. F. (1989). "Quantum States with Einstein–Podolsky–Rosen correlations admitting a hidden-variable model". *W:Physical Review A* **40** (8): 4277–4281. doi:10.1103/PhysRevA.40.4277. PMID 9902666.

nonlocality are inequivalent for any number of parties". *Physical Review Letters* **115** (3). doi:10.1103/PhysRevLett.115.030404. PMID 26230773.

68. Scarani, Valerio (2019). *Bell Nonlocality*. Oxford University Press. p. 8. ISBN 978-0-19-878841-6.
69. Omnès, Roland (1994). *The Interpretation of Quantum Mechanics*. Princeton University Press. pp. 399–400. ISBN 978-0-691-03669-4.
70. Mermin, N. D. (1999). "What Do These Correlations Know About Reality? Nonlocality and the Absurd". *W:Foundations of Physics* **29** (4): 571–587. doi:10.1023/A:1018864225930.
71. Żukowski, Marek (2017). "Bell's Theorem Tells Us Not What Quantum Mechanics is, but What Quantum Mechanics is Not". In Bertlmann, Reinhold; Zeilinger, Anton. *Quantum [Un]Speakables II*. The Frontiers Collection. Cham: Springer International Publishing. pp. 175–185. doi:10.1007/978-3-319-38987-5_10. ISBN 978-3-319-38985-1.

Shor, Peter W.; Smolin, John A.; Wootters, William K. (1999). "Quantum nonlocality without entanglement". *Physical Review A* **59** (2): 1070–1091. doi:10.1103/PhysRevA.59.1070.

73. Haag, Rudolf (1996). *Local Quantum Physics: Fields, Particles, Algebras* (2nd ed.). Springer. pp. 107–108. ISBN 3-540-61451-6.
74. Gurvits, L. (2003). "Classical deterministic complexity of Edmonds' Problem and quantum entanglement". *Proceedings of the thirty-five annual ACM symposium on Theory of computing*. p. 10. doi:10.1145/780542.780545.
75. "Separability of mixed states: necessary and sufficient conditions". *Physics Letters A* **223** (1): 210. 1996. doi:10.1016/S0375-9601(96)00706-2.
76. Pirandola, S. (2015). "Advances in Quantum Teleportation". *Nature Photonics* **9** (10): 641–652. doi:10.1038/nphoton.2015.154.

entanglement swapping: Entangling photons that never interacted". *W:Physical Review Letters* **80** (18): 3891–3894. doi:10.1103/PhysRevLett.80.3891.

78. Bertlmann, Reinhold; Friis, Nicolai (2023-10-05). *Modern Quantum Theory: From Quantum Mechanics to Entanglement and Quantum Information* (in en). Oxford University Press. p. 511. ISBN 978-0-19-150634-5. https://books.google.com/books?id=uzHaEAAAQBAJ&dq=monogamy+of+entanglement&pg=PA511.
79. Grassl, M.; Rötteler, M.; Beth, T. (1998). "Computing local invariants of quantum-bit systems". *Phys. Rev. A* **58** (3): 1833–1839. doi:10.1103/PhysRevA.58.1833.
80. Kraus, Barbara (2010). "Local unitary equivalence of multipartite pure states". *Physical Review Letters* **104** (2). doi:10.1103/PhysRevLett.104.020504. PMID 20366579.
81. Nielsen, M. A. (1999). "Conditions for a Class of Entanglement Transformations". *Physical Review Letters* **83** (2): 436. doi:10.1103/PhysRevLett.83.436.

Dimensionality". *Phys. Rev. Lett.* **111** (6). doi:10.1103/PhysRevLett.111.060502. PMID 23971544.

83. Horodecki, M.; Horodecki, P.; Horodecki, R. (1998). "Mixed-state entanglement and distillation: Is there a *bound* entanglement in nature?". *Phys. Rev. Lett.* **80** (1998): 5239–5242. doi:10.1103/PhysRevLett.80.5239.
84. Wiseman, H. M.; Jones, S. J.; Doherty, A. C. (2007). "Steering, Entanglement, Nonlocality, and the Einstein–Podolsky–Rosen Paradox". *Physical Review Letters* **98** (14). doi:10.1103/PhysRevLett.98.140402. PMID 17501251.
85. Cerf, Nicolas J.; Cleve, Richard (September 1997). "Information-theoretic interpretation of quantum error-correcting codes". *Physical Review A* **56** (3): 1721–1732. doi:10.1103/PhysRevA.56.1721. http://authors.library.caltech.edu/5516/1/CERpra97b.pdf#page=10.
86. Plenio, Martin B.; Virmani, Shashank (2007). "An introduction to entanglement measures". *Quant. Inf. Comp.* **1**: 1–51.

Reviews of Modern Physics **74** (1): 197–234. doi:10.1103/RevModPhys.74.197.

88. Hill, S; Wootters, W. K. (1997). "Entanglement of a Pair of Quantum Bits". *Phys. Rev. Lett.* **78** (26): 5022–5025. doi:10.1103/PhysRevLett.78.5022.
89. Enríquez, M.; Wintrowicz, I.; Życzkowski, K. (March 2016). "Maximally Entangled Multipartite States: A Brief Survey". *Journal of Physics: Conference Series* **698** (1). doi:10.1088/1742-6596/698/1/012003. ISSN 1742-6588.
90. Wang, Yu-Xin; Mu, Liang-Zhu; Vedral, Vlatko; Fan, Heng (17 February 2016). "Entanglement Rényi α entropy". *Physical Review A* **93** (2). doi:10.1103/PhysRevA.93.022324. ISSN 2469-9926. https://link.aps.org/doi/10.1103/PhysRevA.93.022324.
91. Huang, Yichen (21 March 2014). "Computing quantum quantum discord is NP-complete". *New Journal of Physics* **16** (3). doi:10.1088/1367-2630/16/3/033027.

Vacuum State". In Halvorson, Hans. *Deep Beauty: Understanding the Quantum World through Mathematical Innovation*. Cambridge University Press. pp. 317–341. ISBN 978-1-139-49922-4.

93. Jozsa, Richard; Linden, Noah (2002). "On the role of entanglement in quantum computational speed-up". *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **459** (2036): 2011–2032. doi:10.1098/rspa.2002.1097.
94. Renner, R.; Gisin, N.; Kraus, B. (2005). "An information-theoretic security proof for QKD protocols". *Physical Review A* **72**. doi:10.1103/PhysRevA.72.012332.
95. Pirandola, S.; U. L. Andersen; L. Banchi; M. Berta; D. Bunandar; R. Colbeck; D. Englund; T. Gehring *et al*. (2020). "Advances in quantum cryptography". *Adv. Opt. Photon.* **12** (4): 1012–1236. doi:10.1364/AOP.361502.

doi:10.1038/nature.2014.15781. http://www.nature.com/news/entangled-photons-make-a-picture-from-a-paradox-1.15781. Retrieved 13 October 2014.

97. Pearce, Emma; Gemmell, Nathan R.; Flórez, Jefferson; Ding, Jiaye; Oulton, Rupert F.; Clark, Alex S.; Phillips, Chris C. (15 November 2023). "Practical quantum imaging with undetected photons". *Optics Continuum* **2** (11): 2386. doi:10.1364/OPTCON.507154. ISSN 2770-0208. https://opg.optica.org/abstract.cfm?URI=optcon-2-11-2386.
98. Caves, Carlton M.; Fuchs, Christopher A.; Schack, Rüdiger (2002-08-20). "Unknown quantum states: The quantum de Finetti representation". *W:Journal of Mathematical Physics* **43** (9): 4537–4559. doi:10.1063/1.1494471.
99. Chi, Yulin (2022). "A programmable qudit-based quantum processor". *Nature Communications* **13** (1): 1136. doi:10.1038/s41467-022-28767-x. PMID 35246519. PMC 8897515. //www.ncbi.nlm.nih.gov/pmc/articles/PMC8897515/.

5138–5143. doi:10.1103/physreva.47.5138. PMID 9909547. https://ir.library.osaka-u.ac.jp/repo/ouka/all/77656/PhysRevA_47_06_005138.pdf.

101. Wineland, D. J.; Bollinger, J. J.; Itano, W. M.; Moore, F. L.; Heinzen, D. J. (1992). "Spin squeezing and reduced quantum noise in spectroscopy". *Physical Review A* **46** (11): R6797–R6800. doi:10.1103/PhysRevA.46.R6797. PMID 9908086.
102. Holland, M. J; Burnett, K (1993). "Interferometric detection of optical phase shifts at the Heisenberg limit". *Physical Review Letters* **71** (9): 1355–1358. doi:10.1103/PhysRevLett.71.1355. PMID 10055519.
103. Roos, Christian F. (2004). "Control and Measurement of Three-Qubit Entangled States". *Science* **304** (5676): 1478–1480. doi:10.1126/science.1097522. PMID 15178795.
104. Pezzè, L.; Smerzi, A.; Oberthaler, M. K.; Schmied, R.; Treutlein, P. (2018). "Quantum metrology with nonclassical states of atomic ensembles". *Reviews of Modern Physics* **90** (3). doi:10.1103/revmodphys.90.035005.

Review Letters **96** (2): 130501. doi:10.1103/PhysRevLett.96.020501. PMID 16486553.

106. Lee, Hwang; Kok, Pieter; Dowling, Jonathan P. (2002). "A quantum Rosetta stone for interferometry". *Journal of Modern Optics* **49** (14–15): 2325–2338. doi:10.1080/0950034021000011536.
107. Lo Franco, Rosario; Compagno, Giuseppe (14 June 2018). "Indistinguishability of Elementary Systems as a Resource for Quantum Information Processing". *Physical Review Letters* **120** (24). doi:10.1103/PhysRevLett.120.240403. PMID 29957003.
108. Yin, Juan; Cao, Yuan; Li, Yu-Huai; Liao, Sheng-Kai *et al*. (2017). "Satellite-based entanglement distribution over 1200 kilometers". *Science* **356** (6343): 1140–1144. doi:10.1126/science.aan3211. PMID 28619937.
109. Popkin, Gabriel (14 June 2017). "China's quantum satellite achieves 'spooky action' at record distance". *Science*.

Abreu, H. *et al*. (September 2024). "Observation of quantum entanglement with top quarks at the ATLAS detector". *Nature* **633** (8030): 542–547. doi:10.1038/s41586-024-07824-z. ISSN 1476-4687. PMID 39294352. PMC 11410654. //www.ncbi.nlm.nih.gov/pmc/articles/PMC11410654/.

111. "ATLAS achieves highest-energy detection of quantum entanglement". *ATLAS*. 28 September 2023. Retrieved 21 September 2024.
112. "LHC experiments at CERN observe quantum entanglement at the highest energy yet". *CERN*. 18 September 2024. Retrieved 21 September 2024.
113. Afik, Yoav; de Nova, Juan Ramón Muñoz (3 September 2021). "Entanglement and quantum tomography with top quarks at the LHC". *The European Physical Journal Plus* **136** (9): 907. doi:10.1140/epjp/s13360-021-01902-1. ISSN 2190-5444. https://link.springer.com/10.1140/epjp/s13360-021-01902-1.

Entanglement & quantum tomography with top quarks. Retrieved 28 September 2024 – via YouTube.

production in proton–proton collisions at \sqrt{s} = 13~\mathrm{TeV}". *Reports on Progress in Physics* **87** (11). doi:10.1088/1361-6633/ad7e4d. PMID 39527914.

correlation and observation of entanglement in top quark pairs using lepton+jets events from proton-proton collisions at \sqrt{s}=13~\mathrm{TeV}". *Physical Review D* **110** (11). doi:10.1103/PhysRevD.110.112016.
